Supplement: Allocation of Internal Costs

Carl C. Cowen

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1 Allocation of Internal Costs

A profitable business tries to set prices on its products so that all its costs are covered (and a little more added in for profit). Clearly, in order to do that, the company must know what its costs are. If the company makes only one product, the situation is simple, the price on that one product must cover all the costs of the company. Already if the company has two products a problem arises; the prices on the products must be set so that the total costs are met, but that does not help in determining what the prices are on each product. Of course, if a decision is made that makes the price of one of the products high relative to the prices of its competitors, then it will lose business on that product, and thereby might actually not meet its costs.

We should realize that the setting of prices and the *policy* for allocating costs are management decisions, not mathematics problems. Our role in the process is to convert the *policy* for allocating costs into a *mathematical model*, analyze the model mathematically, then report back to management on whether this model reflects their policy decision accurately. To emphasize the role of management, consider the very common grocery marketing practice of selling eggs at (say) 99 cents a dozen when they actually pay (say) \$1.23 a dozen for them. They do this consciously to achieve a certain goal: bring customers into the store who will buy steak at (say) \$10.95 a pound that costs them (say) \$5.67 a pound. As mathematicians, we are not going to try to make this decision! On the other hand, we can advise management on what the costs actually are, facts that should be a part of the marketing decisions.

In addition, we should realize that many of the services that departments perform can be purchased: rather than having a cafeteria in which the food is cooked by company employees, a catering company could be hired to provide meals to employees at a reasonable cost or rather than having the accounting department set up and run retirement and health plans for the employees, the company could hire a benefits administration firm to set up and run these programs. Before a decision can be made on "outsourcing", an accurate assessment of the costs involved in doing the job internally must be made so that they can be compared with the cost to have the job done externally.

> **The Problem**: If a company has several products, how do we allocate the total costs of the company to each of these products so that we know how much each of the products "actually costs" to produce?

Again, we must realize that the actual attribution of the costs is also a management decision: Suppose a company produces forgings and castings and has one janitor to clean up the factory. If the president wishes, the wages of the janitor can all be added to the "cost of producing forgings". Such a decision might make the price of the company's forgings high and the price of its castings low, but that is a *policy* decision. Our role is to help the president carry out this policy, or more likely, help compare the effects of two proposed policies.

One reasonable model of cost allocation is to find out how much effort the janitor spends cleaning up the part of the plant where forgings are produced and how much where castings are produced, then divide the cost of the janitor proportionately. Thus, management will tell us the proportions of effort for the janitor for each of the companies divisions, and we will try to carry out the cost allocation accordingly.

Unfortunately, this simple statement of policy does not solve the problem, for not all parts of the company make products for sale and all parts of the company do use the janitor. For the purposes of the model we will develop, we divide departments of the company into two types, the *service departments* and the *production departments*. The service departments, like the custodial staff, the company cafeteria, the accounting department, supply a service to the departments of the company internally, but do not supply a service or product to customers outside the company. The production departments supply products outside the company, bring in revenue, and it is to these departments that we want to attribute the total costs of the company, including the cost of the service departments. Thus, while the accounting department has costs, like salary, it has no direct customer generated revenue to pay these costs. It writes checks for the production departments, so should get its costs covered by the production departments. A complicating factor is that the accounting department writes checks for itself and for the janitor as well, so the service departments provide service not only to the production departments, but also to the service departments. Similarly, we divide the costs into two categories, the *direct costs* that are paid to someone outside the company, and the *indirect costs* that a department incurs by using the service of one of the service departments. In some sense, the direct costs are "real" and the indirect costs are "funny money". What we really intend is to use the concept of indirect costs to distribute the total direct costs of the company, both those paid by the production departments and those paid by the service departments, among the production departments to arrive at an "actual cost" of producing each product. We will assume that the service departments do not try to make a profit on the work they do.

Example 1 *The Star Chain Company makes two types of chain: roller chain and timing chain. Besides the roller chain and timing chain departments, there is the accounting department, the food services department, and the security and maintenance department. The direct costs of the various departments are 40 (thousand dollars per month) for the accounting department, 60 for the food services department, 100 for the security and maintenance department, 500 for the roller chain department, and 300 for the timing chain department. Thus, the total direct costs for the company are 1000.*

department	*direct costs*	*acct.*	*food*	*sec.*
accounting	*40*	*10%*	*10%*	*10%*
food	*60*	*20%*	*10%*	*10%*
sec. & main.	*100*	*10%*	*10%*	*5%*
roller	*500*	*40%*	*40%*	*45%*
timing	*300*	*20%*	*30%*	*30%*

We want to allocate part of the 1000 to the roller chain department and the rest to the timing chain department in order to determine their "actual costs". The efforts of the various service departments are used by the departments in proportions given in the following table. For example, the 45% in the right column of the table means that 45% of the output of the security and maintenance department is used by the roller chain department. In our model of this problem, we will try to allocate 45% of the total costs of the security and maintenance department to the roller chain department.

Our model will try to allocate the total direct costs of the company among the production departments (in this case, the roller and timing chain departments) by setting up linear equations that express the relationships between the costs of running the various departments. To begin, we set up our notation and give our assumptions:

Suppose an organization has m service departments (numbered 1 to m) and n production departments (numbered $m+1$ to $m+n$). Let d_i denote the direct costs of department i, for $1 \le i \le m+n$, and let c_i denote the total costs of department i, for $1 \le i \le m+n$. That is, c_i is the sum of d_i and the indirect costs allocated to department i due to its use of other departments' services. Let s_{ij} denote the share of the total cost of department j that will be charged to department i, for $1 \le i \le m+n$ and $1 \le j \le m$. Implicit in the notation for the cost sharing, that is, $1 \le j \le m$ rather than $1 \le j \le m+n$, is the assumption that costs of the service departments are shared, the costs for the production departments are not. We further require that $0 \le s_{ij} \le 1$ and that, for each j with $1 \le j \le m$,

$$s_{1j} + s_{2j} + \ldots + s_{m+n,j} = 1$$

This means that the portion of the costs of department j charged to department i are between none ($s_{ij} = 0$) and all ($s_{ij} = 1$) and that all of the costs of department j are charged to some department. Our problem is to determine the total costs of all the departments, that is, we want to determine $c_1, c_2, \ldots, c_{m+n}$.

The intuition behind the choice of variables suggests that the following system of equations should be satisfied:

$$(\dagger) \quad \left\{ \begin{array}{lclclclcccl} c_1 & = & d_1 & + & s_{11}c_1 & + & s_{12}c_2 & + & \cdots & + & s_{1m}c_m \\ c_2 & = & d_2 & + & s_{21}c_1 & + & s_{22}c_2 & + & \cdots & + & s_{2m}c_m \\ & \vdots & & & & & & & & & \\ c_m & = & d_m & + & s_{m1}c_1 & + & s_{m2}c_2 & + & \cdots & + & s_{mm}c_m \\ c_{m+1} & = & d_{m+1} & + & s_{m+1,1}c_1 & + & s_{m+1,2}c_2 & + & \cdots & + & s_{m+1,m}c_m \\ & \vdots & & & & & & & & & \\ c_{m+n} & = & d_{m+n} & + & s_{m+n,1}c_1 & + & s_{m+n,2}c_2 & + & \cdots & + & s_{m+n,m}c_m \end{array} \right.$$

Each equation corresponds to a particular department and asserts that the total costs for that department are its direct costs plus its indirect costs, that is, its share of the total costs of the service departments who provide services to it. Since the direct costs (d_i) and the proportions (s_{ij}) of costs to be allocated are assumed to be known, this is a system of $m+n$ equations in $m+n$ unknowns (c_i). Examination of the system, however, shows that actually, the first m equations involve only the first m unknowns and the last n equations give the last n unknowns (the production department costs) in terms of the first m unknowns (the service department costs).

EXAMPLE 1, continued. For the Star Chain Company, numbering the departments in the order accounting, food, security, roller chain, timing chain; these equations become

$$\left\{ \begin{array}{lcrcrcrcr} c_1 & = & 40 & + & .1c_1 & + & .1c_2 & + & .1c_3 \\ c_2 & = & 60 & + & .2c_1 & + & .1c_2 & + & .1c_3 \\ c_3 & = & 100 & + & .1c_1 & + & .1c_2 & + & .05c_3 \\ c_4 & = & 500 & + & .4c_1 & + & .4c_2 & + & .45c_3 \\ c_5 & = & 300 & + & .2c_1 & + & .3c_2 & + & .3c_3 \end{array} \right.$$

Rewriting the first three equations, we get

$$\begin{cases} .9c_1 & - & .1c_2 & - & .1c_3 & = & 40 \\ -.2c_1 & + & .9c_2 & - & .1c_3 & = & 60 \\ -.1c_1 & - & .1c_2 & + & .95c_3 & = & 100 \end{cases}$$

which has the solution $c_1 = 68.68$, $c_2 = 95.54$, and $c_3 = 122.55$. Putting these values into the last two equations, we find that the total costs of the roller chain department are $c_4 = 620.84$ and total costs of the timing chain department are $c_5 = 379.16$. For this example, the combined total costs of the production departments is 1000 which is also the combined direct costs for the entire company, so we have achieved our goal of allocating all the direct costs among the production departments. ■

Was this balance a lucky accident or will it always happen this way? Are there similar problems in which some of the costs are negative? These are some of the questions we must answer if we are to evaluate our model. Clearly, if the balance between the total costs allocated to production and the total direct costs is an accident, then the model does not achieve its stated goal. Even if the balance always occurs, if sometimes the costs come out negative, we will lose faith in the model because a department should not get useful output for (less than) nothing. In fact these things are not accidental, we will analyze the model and then prove a theorem that says, under some reasonable conditions, all costs are positive and the company's total direct costs are allocated among the production departments.

The system of equations (†) will be easier to understand if we express it in terms of block matrices. Let $C_s = (c_1, c_2, \ldots, c_m)$ be the column vector of total costs of the service departments and let $C_p = (c_{m+1}, c_{m+2}, \ldots, c_{m+n})$ be the column vector of total costs of the production departments. Similarly, let D_s and D_p be the column vectors of direct costs of the service and production departments and let S_s and S_p be the $m \times m$ and $n \times m$ matrices whose entries are the proportions of the costs of the service departments to be allocated to the service and production departments. With this notation, the system (†) becomes

$$(\ddagger) \qquad \begin{pmatrix} C_s \\ C_p \end{pmatrix} = \begin{pmatrix} D_s \\ D_p \end{pmatrix} + \begin{pmatrix} S_s \\ S_p \end{pmatrix} C_s$$

Isolating the part of the system involving only the service departments, we get the equation

$$C_s = D_s + S_s C_s$$

In other words,

$$\begin{aligned} C_s - S_s C_s &= D_s \\ (I - S_s)C_s &= D_s \end{aligned}$$

and if $I - S_s$ is invertible,

$$C_s = (I - S_s)^{-1} D_s$$

This implies that

$$C_p = D_p + S_p (I - S_s)^{-1} D_s$$

so the system is solved.

It is clear from this calculation that the applicability of the model will depend on the existence and nature of $(I - S_s)^{-1}$. We are now ready to state and prove the theorem justifying the model, and it is really a theorem about $(I - S_s)^{-1}$.

Theorem 2 *Let the notation be as above. Suppose*

(a) for each i, we have $d_i \geq 0$,

(b) for each i and j, $s_{i,j} \geq 0$

and

(c) for each j, we have $\sum_{i=1}^{m+n} s_{i,j} = 1$, and there is at least one i with $m+1 \leq i \leq m+n$ for which $s_{ij} > 0$.

Then

(1) $(I - S_s)^{-1}$ exists and has all non-negative entries,

(2) $C_s = (I - S_s)^{-1} D_s$ and $C_p = D_p + S_p(I - S_s)^{-1} D_s$, is the unique solution of the system of equations (†),

(3) for each i, we have $c_i \geq 0$, and

(4) $\displaystyle\sum_{i=m+1}^{m+n} c_i = \sum_{i=1}^{m+n} d_i$

Before proving the theorem, let's examine the meaning of each of the statements. The assumption (a) is that all the direct costs are non-negative; this is clearly a sensible assumption since, as noted above, we cannot expect to get something for (less than) nothing. Assumption (b) says that every service department should provide service to some production department. This assumption makes sense in that the focus of the company is on the production departments, so a service department should provide service to one of them. On the other hand, it is easy to imagine that it would be convenient to have departments in a company that served production by serving the service departments, for example, it is conceivable that a company might have a data processing department that served only the accounting and inventory departments and no production departments. In this case, the theorem above would not (directly) apply. We have two choices: we could create another model (and another theorem) in which there were three classes of departments, production, departments serving production departments and service departments not serving production departments, or, more easily, we could incorporate, for the purposes of the model only, departments not directly serving production departments into those that were. For example, we could, for the purposes of analysis only, incorporate the data processing and the accounting departments, and the new (fictitious) department would then fit into the current model.

Parts (1) and (2) of the conclusion reiterate the formulas given above, assert that they make mathematical sense, and give further information about them. Part (3) of the conclusion says that the computed costs satisfy the minimum demand of reasonableness, that they be non-negative. Part (4) says that the model presented solves the problem: it asserts that the sum of the total costs of the production departments as computed by the model equal the total direct costs of the company.

Proof. The definition of the matrix $\begin{pmatrix} S_s \\ S_p \end{pmatrix}$ shows that the sum of the entries in each column is 1 and hypothesis (b) of the theorem says that each column of S_p has a positive entry. It follows that $||S_s||_1 < 1$. Applying Theorem 5.11 of *Linear Algebra for Engineering and Science* with $\lambda = 1$, and $p = 1$, we find that $(I - S_s)^{-1}$ exists and in fact

$$(I - S_s)^{-1} = \sum_{k=0}^{\infty} S_s^k$$

Since S_s has all non-negative entries, S_s^k has non-negative entries for every k, and since each term of the series does, it follows that $(I - S_s)^{-1}$ has all non-negative entries. This proves (1).

Since $(I - S_s)^{-1}$ exists, the derivation of the formulas $C_s = (I - S_s)^{-1}D_s$ and $C_p = D_p + S_p(I - S_s)^{-1}D_s$ given above is correct, and we see that each C_s and C_p is a sum of products of matrices with non-negative entries, so each has non-negative entries, that is, $c_i \geq 0$ for each i. This proves (3).

To complete the proof, we need some auxiliary notation: for each positive integer k, let t_k be the $1 \times k$ matrix of ones, $t_k = (1\ 1\ \cdots\ 1)$. If v is a $k \times 1$ matrix, then $t_k v = v_1 + v_2 + \ldots + v_k$.

As above (‡), we have

$$\begin{pmatrix} C_s \\ C_p \end{pmatrix} = \begin{pmatrix} D_s \\ D_p \end{pmatrix} + \begin{pmatrix} S_s \\ S_p \end{pmatrix} C_s$$

Therefore,

$$(t_m\ t_n)\begin{pmatrix} C_s \\ C_p \end{pmatrix} = (t_m\ t_n)\begin{pmatrix} D_s \\ D_p \end{pmatrix} + \left((t_m\ t_n)\begin{pmatrix} S_s \\ S_p \end{pmatrix}\right) C_s$$

Since $\begin{pmatrix} S_s \\ S_p \end{pmatrix}$ has m columns and each column sum is equal to 1 it follows that

$$(t_m\ t_n)\begin{pmatrix} S_s \\ S_p \end{pmatrix} = t_m$$

Completing the multiplication, we obtain

$$t_m C_s + t_n C_p = t_m D_s + t_n D_p + t_m C_s$$

which means

$$t_n C_p = t_m D_s + t_n D_p$$

but this is just conclusion (4):

$$\sum_{i=m+1}^{m+n} c_i = \sum_{i=1}^{m+n} d_i$$

■

Exercises

1. The Acme Manufacturing Company makes molded plastic products. They have divided their organization into 6 departments: the toy department, the kitchen gadget department, and the novelty department, which are production departments, and the accounting department, the maintenance department, and the security department, which are service departments. Their activities and costs are shown in the table below. Direct costs are given in thousands of dollars and the "20%" in the upper right corner of the table means that 20% of the security departments efforts are providing service to the accounting department.

department	direct costs	acct.	main.	sec.
accounting	20	10%	10%	20%
maintenance	100	10%	20%	10%
security	40	10%	10%	0%
kitchen	50	20%	20%	30%
novelty	20	20%	20%	20%
toy	100	30%	20%	20%

   (a) Find the total costs for each department and verify that the combined total costs of the production departments are equal to the total direct costs for all departments.

   (b) Al's Protection Company (A. Capone, president) has offered their services to Acme for $60,000 per year. Should the company eliminate their security department and buy protection from Al's company instead? Why?

2. Yolanda, a chemical engineer, and Zeke, an industrial engineer, share an office, a secretary, and a computer programmer in their consulting businesses. The secretary is paid $8 an hour and works 25 percent of the time for the computer programmer, 30 percent of the time for Yolanda, and 45 percent of the time for Zeke. The computer programmer is paid $13 an hour and works 20 percent of the time for the secretary, 20 percent of the time doing maintenance on her own programs, 35 percent of the time for Yolanda, and 25 percent of the time for Zeke. They split the other office expenses equally, each paying $3 an hour. If Yolanda wants to earn a salary of $30 an hour and Zeke wants to earn $25 an hour, how much should they each charge their customers?

3. The Top Brass Manufacturing Company makes brass clothing accessories. They have divided their organization into 4 departments: the button department and the buckle department, which are production departments, and the personnel department and the accounting department, which are service departments. Their activities and costs

department	direct costs	pers.	acct.
personnel	30	40%	40%
accounting	20	20%	20%
button	50	30%	20%
buckle	40	10%	20%

are shown in the table above. Direct costs are given in thousands of dollars and the "40%" in the upper right corner of the table means that 40% of the accounting department's efforts are providing service to the personnel department. Find the total costs for each department.

4. Pinkham Pharmaceuticals makes and sells penicillin, potassium chloride (KCl) in aqueous solution, and Lydia Pinkham's Pink Pills for Pale People. In addition, the company consists of the production services department, the accounting department, the advertising department, and the product development department. Ms. Pinkham, CEO, has found that the direct costs of the penicillin department are 600 (thousand dollars per month); the potassium chloride department, 240; the pink pill department, 180; the production services department, 300; the accounting department, 120; the advertising department, 500; and the development department, 280. The following table represents use of the various departments services by the other departments; she assigned the costs of the advertising and development departments not on the basis of reports from her accounting staff, rather on the basis of her expectation of the benefits from that department's operation. Find the total costs for each department.

department	prod.	acct.	adv.	dev.
penicillin	40%	10%	20%	30%
KCl	20%	15%	30%	20%
pills	10%	30%	45%	10%
prod. ser.	10%	10%	0%	30%
accounting	5%	5%	0%	0%
advert.	5%	20%	0%	0%
develop.	10%	10%	5%	10%

5. The Boilermaker Iron Company makes assorted castings and forgings. They have divided their organization into 4 departments: the casting department, the forging department, the accounting department, and the maintenance department. Some of their activities and costs are shown in the table below.

department	direct costs	acct.	main.
accounting	50	10%	10%
maintenance	100	20%	10%
casting	750	60%	30%
forging	300	10%	50%

Unfortunately, this company does not directly fit our model because the casting department does 2/3 of its work for outside customers but does 1/3 of its work making castings for the forging department, in other words, it is neither a purely production department nor a purely service department. Modify the model or the way you set the problem up, explaining what you are doing, and find the total costs that should be passed on to the customers of the forging department and the outside customers of the casting department.