Math S165 Friday Work

Axioms for the Real Numbers and Their Basic Properties

In some sense, we are familiar with the real numbers and their basic properties. The lists below incorporate enough of their basic properties as axioms that we can characterize the real numbers and develop all the other properties from these. The items below begin to do that.

In addition, we are familiar with the natural numbers, $\mathbb{N}$, the integers, $\mathbb{Z}$, and the rational numbers, $\mathbb{Q}$, (which are quotients of integers) and their properties. Unlike the real numbers, we will not try to develop their properties from axioms, although we could do so.

We assume there is a set, $\mathbb{R}$, the *set of real numbers*, that satisfies the following axioms:

Algebraic properties

Axiom 1. $a + b = b + a$ for all a and b in $\mathbb{R}$.
Axiom 2. $(a + b) + c = a + (b + c)$ for all a, b, and c in $\mathbb{R}$.
Axiom 3. There exists an element 0 in $\mathbb{R}$ such that $a + 0 = 0 + a = a$ for all a in $\mathbb{R}$.
Axiom 4. For each a in $\mathbb{R}$, there is an element $-a$ in $\mathbb{R}$ such that $a + (-a) = (-a) + a = 0$.
Axiom 5. $ab = ba$ for all a and b in $\mathbb{R}$.
Axiom 6. $(ab)c = a(bc)$ for all a, b, and c in $\mathbb{R}$.
Axiom 7. There exists an element 1 in $\mathbb{R}$ such that $1 \neq 0$ and $a \cdot 1 = 1 \cdot a = a$ for all a in $\mathbb{R}$.
Axiom 8. For each a in $\mathbb{R}$, $a \neq 0$, there is an element a^{-1} in $\mathbb{R}$ such that

$$a \cdot a^{-1} = a^{-1} \cdot a = 1$$

Axiom 9. $(a + b)c = ac + bc$ and $a(b + c) = ab + ac$ for all a, b, and c in $\mathbb{R}$.

1. If a, b, and c are real numbers such that $a + b = a + c$, then $b = c$.

2. If a, b, and c are real numbers such that $a \neq 0$ and $ab = ac$, then $b = c$.

3. Corollary: If a and d are real numbers such that $a + d = 0$, then $d = -a$. Similarly, if a and e are real numbers with $a \neq 0$ such that $ae = 1$, then $e = a^{-1}$.

4. Corollary: If z and a are real numbers such that $z + a = a$, then $z = 0$. Similarly, if u and a are real numbers with $a \neq 0$ such that $ua = a$, then $u = 1$.

Statements 3 and 4 say that the numbers mentioned in Axioms 3, 4, 7, and 8 are unique. This allows us to use the article 'the' (rather than 'a' or 'an') in the definitions in the following sentences. The numbers 0 and 1 are called the *additive identity* and the *multiplicative identity* respectively. For a a real number, $-a$ is called the *additive inverse of* a and when $a \neq 0$, the number a^{-1} is called the *multiplicative inverse of* a.

5. If a is any real number, $a \cdot 0 = 0$.

6. If a is any real number, $(-1) \cdot a = -a$.

7. If a is any real number, $-(-a) = a$.

8. $(-1) \cdot (-1) = 1$.

9. If a is any non-zero real number, $(a^{-1})^{-1} = a$.

10. If a and b are real numbers such that $a \cdot b = 0$, then either $a = 0$ or $b = 0$.

Definition If a and b are real numbers, we define $a - b$ by $a - b = a + (-b)$.

Definition If a and b are real numbers with $b \neq 0$, we define a/b by $a/b = a \cdot b^{-1}$.

11. If a, b, and x are real numbers such that $x + b = a$, then $x = a - b$. Similarly, if a, b, and x are real numbers, with $b \neq 0$, such that $bx = a$, then $x = a/b$.

Order properties

There is a set $\mathbf{P}$ of real numbers, called the *positive real numbers*, such that

Axiom 10. If a and b are in $\mathbf{P}$, then $a + b$ is in $\mathbf{P}$.
Axiom 11. If a and b are in $\mathbf{P}$, then $a \cdot b$ is in $\mathbf{P}$.
Axiom 12. For each a in $\mathbb{R}$, exactly one of the following is true:

$$a \in \mathbf{P} \quad \text{or} \quad a = 0 \quad \text{or} \quad (-a) \in \mathbf{P}$$

Definition We say x *is less than* y or y *is greater than* x and write $x < y$ or $y > x$ if $y - x$ is in $\mathbf{P}$.

Definition We say x *is less than or equal to* y or y *is greater than or equal to* x and write $x \leq y$ or $y \geq x$ if either $x = y$ or $y - x$ is in $\mathbf{P}$.

12. If a, b, and c are real numbers such that $a < b$ and $b < c$, then $a < c$.

13. If a and b are real numbers, then exactly one of the following holds:

$$a < b \quad \text{or} \quad a = b \quad \text{or} \quad a > b$$

14. If a and b are real numbers such that $a \leq b$ and $b \leq a$, then $a = b$.

15. If a and b are real numbers such that $a > b$, then $-a < -b$. In particular, if $a > 0$, then $-a < 0$.

16. If a is a non-zero real number, then $a^2 > 0$.

17. $1 > 0$.

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18. If a, b, and c are real numbers such that $a < b$, then $a + c < b + c$.

19. If a, b, c, and d are real numbers such that $a < b$ and $c < d$, then $a + c < b + d$.

20. If a, b, and c are real numbers such that $a < b$ and $c > 0$, then $ac < bc$.

21. If a, b, and c are real numbers such that $a < b$ and $c < 0$, then $ac > bc$.

22. Suppose d and e are real numbers. If $d > 0$ and $e > 0$, then $de > 0$. If $d > 0$ and $e < 0$, then $de < 0$. If $d < 0$ and $e < 0$, then $de > 0$.

23. If n is a natural number, then $n > 0$.

24. If $a > 0$, then $1/a > 0$ also. In particular, if n is a natural number, $1/n > 0$.

25. If a and b are real numbers such that $a > b$, then $a > \frac{1}{2}(a + b) > b$.

26. If a is a positive real number, that is, $a > 0$, then $a > \frac{1}{2}a > 0$, so there is no smallest positive real number.

27. If a is a real number such that $0 \leq a < \varepsilon$ for every positive number ε, then $a = 0$.

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Definition Suppose S is a non-empty set of real numbers. A real number b is said to be *an upper bound for* S if $x \leq b$ for every x in S. A real number c is said to be *a lower bound for* S if $x \geq c$ for every x in S.

Definition Suppose S is a non-empty set of real numbers. We say S is *bounded above* if there is a real number b such that b is an upper bound for S. We say S is *bounded below* if there is a real number c such that c is a lower bound for S. We say S is *bounded* if S is both bounded above and bounded below. The set S is called *unbounded* if either it is not bounded above or not bounded below.

28. Suppose S is a non-empty set of real numbers. Suppose that b is an upper bound for S and c is a lower bound for S. Prove that $b \geq c$.

29. Let S, b, an c be as in Exercise 28. Is it always true that $b > c$? (That is, is it true for all sets S of real numbers?)

30. Let $S = \left\{ \dfrac{2r}{3+r^2} : r \geq 1 \right\}$. Prove that S is bounded.

31. Let S and T be non-empty sets of real numbers and suppose that both S and T are bounded. Show that $S \cup T$ is bounded also.

32. Let S and T be non-empty sets of real numbers such that $S \subset T$. Suppose T is a bounded set. Show that S is also bounded.

Definition Suppose S is a non-empty set of real numbers. A real number a is said to be a *maximum of* S if

1) a is an element of S

   and

2) if $a \geq x$ for all x in S.

   Similarly,

Definition Suppose S is a non-empty set of real numbers. A real number z is said to be a *minimum of* S if

1) z is an element of S

   and

2) $z \leq x$ for all x in S.

33. Suppose S is a non-empty set for which a_1 and a_2 are both maxima of S. Prove that $a_1 = a_2$. That is, if S has a maximum, it has only one and we write $a_1 = \max S$.

34. Suppose S is a non-empty set whose maximum is a. Let $T = \{x : -x \in S\}$. Show that T has a minimum.

35. Give an example of a bounded set with no maximum and no minimum.

Definition Suppose S is a non-empty set of real numbers that is bounded above. A real number u is said to be a *supremum of S* or a *least upper bound for S* if

1) u is an upper bound for S (that is, $x \leq u$ for every x in S)

   and

2) if b is any other upper bound for S, $u \leq b$.

   Similarly,

Definition Suppose S is a non-empty set of real numbers that is bounded below. A real number w is said to be an *infimum of S* or a *greatest lower bound for S* if

1) w is a lower bound for S (that is, $x \geq w$ for every x in S)

   and

2) if c is any other lower bound for S, $w \geq c$.

36. Suppose S is a non-empty set for which b_1 and b_2 are both suprema of S. Prove that $b_1 = b_2$. That is, if S has a supremum, it has only one and we write $b_1 = \sup S$.

37. Suppose S is a non-empty set whose supremum is b. Let $T = \{x : -x \in S\}$. Find the infimum of T and show that it is the infimum.

38. Suppose S and T are bounded sets with $b = \sup S$ and $c = \sup T$. Find the supremum of $S \cup T$ and show that it is the supremum.

39. Suppose S and T are bounded sets with $b = \sup S$ and $c = \sup T$. Let V be the set defined by $V = \{z : z = x + y \text{ for } x \in S \text{ and } y \in T\}$. ($V$ is often written as $S + T$.) Find the supremum of $S + T$ and show that it is the supremum.

40. Let S and T be bounded sets. Let P be the set $P = \{q : q = xy \text{ for } x \in S \text{ and } y \in T\}$. Find the supremum of P and show that it is the supremum.

The Completeness Axiom

Axiom 13. Every non-empty set S of real numbers that is bounded above has a supremum; that is, there is a real number b so that $b = \sup S$.

41. Let S be a non-empty set that is bounded above, and let $b = \sup S$. Show that if $\epsilon > 0$, then there is x in S so that $x > b - \epsilon$.

42. Show that every non-empty set T of real numbers that is bounded below has an infimum; that is, there is a real number c so that $c = \inf T$.

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43. Show that the set $\mathbb{N} = \{1, 2, 3, 4, \cdots\}$ of positive integers is unbounded above.

44. Suppose x is a real number. Show that there is a positive integer n so that $n > x$.

45. Prove: if $x > 0$ and y is any real number, there is a positive integer n so that $nx > y$.

46. Let a be a positive real number. Show that if $h > 0$, there is a positive integer n so that

$$\frac{a}{n} < h$$

47. Let a and b be real numbers so that $0 < a < b$. Show that there is a rational number between a and b, that is, show that there are positive integers p and q so that

$$a < \frac{p}{q} < b$$

48. Suppose the non-negative real numbers a, c, and x satisfy

$$a \leq c \leq a + \frac{x}{n}$$

for every positive integer n. Show that $c = a$.

49. Suppose S and T are non-empty sets of real numbers such that for each element s of S and each element t of T, then $s \leq t$.
Prove that S has a supremum and T has an infimum and that they satisfy

$$\sup S \leq \inf T$$

50. Let a be a positive real number. Show that there is a positive real number r so that $r^2 = a$.

51. Let a be a positive real number. Show that the positive real number r in problem 50 is unique, that is, show that if r and s are positive real numbers so that $r^2 = a$ and $s^2 = a$, then $r = s$.

We showed that there is no rational number q such that $q^2 = 2$. However, problem 50 shows that there is a positive *real* number r such that $r^2 = 2$. Since the number is unique, we are justified in giving it a name: we call it 'the square root of 2' and write $\sqrt{2}$ for that number. More generally, if a is any positive real number, we write $\sqrt{a}$ for the unique positive real number that satisfies $(\sqrt{a})^2 = a$.

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Recall that if a and b are real numbers with $a < b$, the *open interval* (a, b) is the set $(a, b) = \{x : a < x < b\}$ and the *closed interval* $[a, b]$ is the set $[a, b] = \{x : a \le x \le b\}$.

Definition A subset S of the real numbers is said to be *open* or *an open set* if every number x in S is in an open interval (a, b) such that $(a, b) \subset S$, that is, S is open if for each x in S there are numbers a and b with $a < x < b$ such that $(a, b) \subset S$.

Definition A subset K of the real numbers is said to be *closed* or *a closed set* if the complement of K, that is, $K^c = \mathbb{R} \setminus K$ is open.

Note that the term 'open interval' is consistent with the definition of 'open set' because if x is a number in $S = (a, b)$, then $a < x < b$ and $(a, b) \subset S$.

52. Show that the set of positive real numbers is open.

53. Show that interval $[2, 5) = \{s : 2 \le s < 5\}$ is not open.

54. Show that if S is an open set and if T is an open set, then $S \cup T$ is an open set.

55. Show that if S is an open set and if T is an open set, then $S \cap T$ is an open set.

56. Show that a closed interval is a closed set.

57. Suppose $\mathcal{I}$ is a set (we will call $\mathcal{I}$ an index set). Show that if S_α is an open set for each α in $\mathcal{I}$, the union of all the S_α's

$$\bigcup_{\alpha \in \mathcal{I}} S_\alpha$$

is open.

58. Suppose n is a positive integer and suppose S_j is an open set for each j, with $j = 1, 2, 3, \cdots, n$. Prove by induction that the $\bigcap_{j=1}^{n} S_j$ is an open set.

59. Show that the result of Problem 58 cannot be improved: find an infinite number of open sets, S_1, S_2, S_3, $\cdots$ such that the closed interval $[0, 1]$ is their intersection: $\bigcap_{j=1}^{\infty} S_j = [0, 1]$.