CHAPTER 1

PRELIMINARIES

In this initial chapter we will present the background needed for the study of real analysis. Section 1.1 consists of a brief survey of set operations and functions, two vital tools for all of mathematics. In it we establish the notation and state the basic definitions and properties that will be used throughout the book. We will regard the word "set" as synonymous with the words "class", "collection", and "family", and we will not define these terms or give a list of axioms for set theory. This approach, often referred to as "naive" set theory, is quite adequate for working with sets in the context of real analysis.

Section 1.2 is concerned with a special method of proof called Mathematical Induction. It is related to the fundamental properties of the natural number system and, though it is restricted to proving particular types of statements, it is important and used frequently. An informal discussion of the different types of proofs that are used in mathematics, such as contrapositives and proofs by contradiction, can be found in Appendix A.

In Section 1.3 we apply some of the tools presented in the first two sections of this chapter to a discussion of what it means for a set to be finite or infinite. Careful definitions are given and some basic consequences of these definitions are derived. The important result that the set of rational numbers is countably infinite is established.

In addition to introducing basic concepts and establishing terminology and notation, this chapter also provides the reader with some initial experience in working with precise definitions and writing proofs. The careful study of real analysis unavoidably entails the reading and writing of proofs, and like any skill, it is necessary to practice. This chapter is a starting point.

Section 1.1 Sets and Functions

To the reader: In this section we give a brief review of the terminology and notation that will be used in this text. We suggest that you look through quickly and come back later when you need to recall the meaning of a term or a symbol.

If an element x is in a set A, we write

$$x \in A$$

and say that x is a **member** of A, or that x **belongs** to A. If x is *not* in A, we write

$$x \notin A.$$

If every element of a set A also belongs to a set B, we say that A is a **subset** of B and write

$$A \subseteq B \quad \text{or} \quad B \supseteq A$$

We say that a set A is a **proper subset** of a set B if $A \subseteq B$, but there is at least one element of B that is not in A. In this case we sometimes write

$$A \subset B.$$

1.1.1 Definition Two sets A and B are said to be **equal**, and we write $A = B$, if they contain the same elements.

Thus, to prove that the sets A and B are equal, we must show that

$$A \subseteq B \quad \text{and} \quad B \subseteq A.$$

A set is normally defined by either listing its elements explicitly, or by specifying a property that determines the elements of the set. If P denotes a property that is meaningful and unambiguous for elements of a set S, then we write

$$\{x \in S : P(x)\}$$

for the set of all elements x in S for which the property P is true. If the set S is understood from the context, then it is often omitted in this notation.

Several special sets are used throughout this book, and they are denoted by standard symbols. (We will use the symbol $:=$ to mean that the symbol on the left is being *defined* by the symbol on the right.)

- The set of **natural numbers** $\mathbb{N} := \{1, 2, 3, \cdots\}$,
- The set of **integers** $\mathbb{Z} := \{0, 1, -1, 2, -2, \cdots\}$,
- The set of **rational numbers** $\mathbb{Q} := \{m/n : m, n \in \mathbb{Z} \text{ and } n \neq 0\}$,
- The set of **real numbers** $\mathbb{R}$.

The set $\mathbb{R}$ of real numbers is of fundamental importance for us and will be discussed at length in Chapter 2.

1.1.2 Examples **(a)** The set

$$\{x \in \mathbb{N} : x^2 - 3x + 2 = 0\}$$

consists of those natural numbers satisfying the stated equation. Since the only solutions of this quadratic equation are $x = 1$ and $x = 2$, we can denote this set more simply by $\{1, 2\}$.

(b) A natural number n is **even** if it has the form $n = 2k$ for some $k \in \mathbb{N}$. The set of even natural numbers can be written

$$\{2k : k \in \mathbb{N}\},$$

which is less cumbersome than $\{n \in \mathbb{N} : n = 2k,\ k \in \mathbb{N}\}$. Similarly, the set of **odd** natural numbers can be written

$$\{2k - 1 : k \in \mathbb{N}\}. \qquad \square$$

Set Operations

We now define the methods of obtaining new sets from given ones. Note that these set operations are based on the meaning of the words "or", "and", and "not". For the union, it is important to be aware of the fact that the word "or" is used in the *inclusive sense*, allowing the possibility that x may belong to both sets. In legal terminology, this inclusive sense is sometimes indicated by "and/or".

1.1.3 Definition **(a)** The **union** of sets A and B is the set

$$A \cup B := \{x : x \in A \text{ or } x \in B\}.$$

(b) The **intersection** of the sets A and B is the set

$$A \cap B := \{x : x \in A \text{ and } x \in B\}.$$

(c) The **complement of** B **relative to** A is the set

$$A \backslash B := \{x : x \in A \text{ and } x \notin B\}.$$

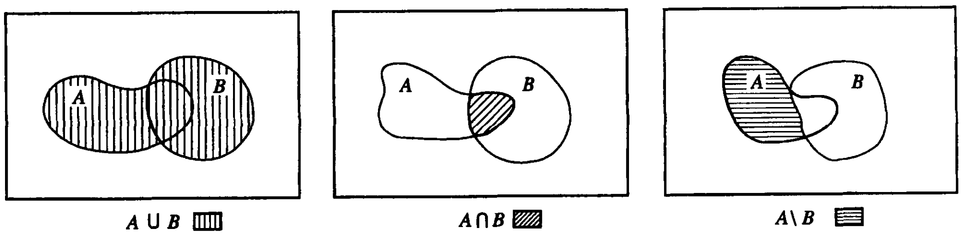

Figure 1.1.1 (a) $A \cup B$ (b) $A \cap B$ (c) $A \backslash B$

The set that has no elements is called the **empty set** and is denoted by the symbol $\emptyset$. Two sets A and B are said to be **disjoint** if they have no elements in common; this can be expressed by writing $A \cap B = \emptyset$.

To illustrate the method of proving set equalities, we will next establish one of the *DeMorgan laws* for three sets. The proof of the other one is left as an exercise.

1.1.4 Theorem *If A, B, C are sets, then*

(a) $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$,

(b) $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$.

Proof. To prove (a), we will show that every element in $A \backslash (B \cup C)$ is contained in both $(A \backslash B)$ and $(A \backslash C)$, and conversely.

If x is in $A \backslash (B \cup C)$, then x is in A, but x is not in $B \cup C$. Hence x is in A, but x is neither in B nor in C. Therefore, x is in A but not B, and x is in A but not C. Thus, $x \in A \backslash B$ and $x \in A \backslash C$, which shows that $x \in (A \backslash B) \cap (A \backslash C)$.

Conversely, if $x \in (A \backslash B) \cap (A \backslash C)$, then $x \in (A \backslash B)$ and $x \in (A \backslash C)$. Hence $x \in A$ and both $x \notin B$ and $x \notin C$. Therefore, $x \in A$ and $x \notin (B \cup C)$, so that $x \in A \backslash (B \cup C)$.

Since the sets $(A \backslash B) \cap (A \backslash C)$ and $A \backslash (B \cup C)$ contain the same elements, they are equal by Definition 1.1.1. Q.E.D.

There are times when it is desirable to form unions and intersections of more than two sets. For a finite collection of sets $\{A_1, A_2, \cdots, A_n\}$, their union is the set A consisting of all elements that belong to *at least one* of the sets A_k, and their intersection consists of all elements that belong to *all* of the sets A_k.

This is extended to an infinite collection of sets $\{A_1, A_2, \cdots, A_n, \cdots\}$ as follows. Their **union** is the set of elements that belong to *at least one* of the sets A_n. In this case we write

$$\bigcup_{n=1}^{\infty} A_n := \{x : x \in A_n \text{ for some } n \in \mathbb{N}\}.$$

Similarly, their **intersection** is the set of elements that belong to *all* of these sets A_n. In this case we write

$$\bigcap_{n=1}^{\infty} A_n := \{x : x \in A_n \text{ for all } n \in \mathbb{N}\}.$$

Cartesian Products

In order to discuss functions, we define the Cartesian product of two sets.

1.1.5 Definition If A and B are nonempty sets, then the **Cartesian product** $A \times B$ of A and B is the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$. That is,

$$A \times B := \{(a, b) : a \in A, b \in B\}.$$

Thus if $A = \{1, 2, 3\}$ and $B = \{1, 5\}$, then the set $A \times B$ is the set whose elements are the ordered pairs

$$(1, 1), \quad (1, 5), \quad (2, 1), \quad (2, 5), \quad (3, 1), \quad (3, 5).$$

We may visualize the set $A \times B$ as the set of six points in the plane with the coordinates that we have just listed.

We often draw a diagram (such as Figure 1.1.2) to indicate the Cartesian product of two sets A and B. However, it should be realized that this diagram may be a simplification. For example, if $A := \{x \in \mathbb{R} : 1 \le x \le 2\}$ and $B := \{y \in \mathbb{R} : 0 \le y \le 1 \text{ or } 2 \le y \le 3\}$, then instead of a rectangle, we should have a drawing such as Figure 1.1.3.

We will now discuss the fundamental notion of a *function* or a *mapping*.

To the mathematician of the early nineteenth century, the word "function" meant a definite formula, such as $f(x) := x^2 + 3x - 5$, which associates to each real number x another number $f(x)$. (Here, $f(0) = -5$, $f(1) = -1$, $f(5) = 35$.) This understanding excluded the case of different formulas on different intervals, so that functions could not be defined "in pieces".

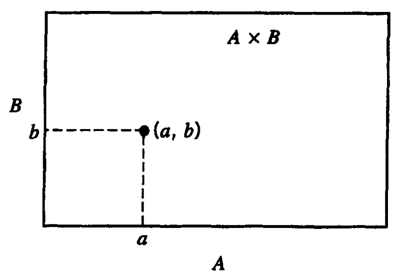

Figure 1.1.2

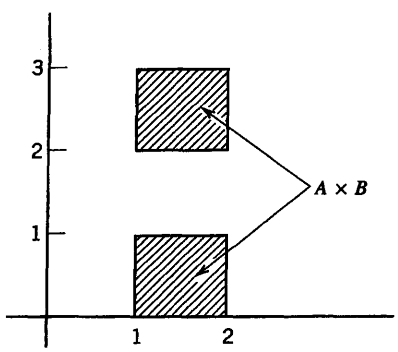

Figure 1.1.3

As mathematics developed, it became clear that a more general definition of "function" would be useful. It also became evident that it is important to make a clear distinction between the function itself and the values of the function. A revised definition might be:

> A function f from a set A into a set B is a rule of correspondence that assigns to each element x in A a uniquely determined element $f(x)$ in B.

But however suggestive this revised definition might be, there is the difficulty of interpreting the phrase "rule of correspondence". In order to clarify this, we will express the definition entirely in terms of sets; in effect, we will define a function to be its **graph**. While this has the disadvantage of being somewhat artificial, it has the advantage of being unambiguous and clearer.

1.1.6 Definition Let A and B be sets. Then a **function** from A to B is a set f of ordered pairs in $A \times B$ such that for each $a \in A$ there exists a unique $b \in B$ with $(a, b) \in f$. (In other words, if $(a, b) \in f$ and $(a, b') \in f$, then $b = b'$.)

The set A of first elements of a function f is called the **domain** of f and is often denoted by $D(f)$. The set of all second elements in f is called the **range** of f and is often denoted by $R(f)$. Note that, although $D(f) = A$, we only have $R(f) \subseteq B$. (See Figure 1.1.4.)

The essential condition that:

$$(a, b) \in f \qquad \text{and} \qquad (a, b') \in f \qquad \text{implies that} \qquad b = b'$$

is sometimes called the *vertical line test.* In geometrical terms it says every vertical line $x = a$ with $a \in A$ intersects the graph of f exactly once.

The notation

$$f : A \to B$$

is often used to indicate that f is a function from A into B. We will also say that f is a **mapping** of A into B, or that f **maps** A into B. If (a, b) is an element in f, it is customary to write

$$b = f(a) \qquad \text{or sometimes} \qquad a \mapsto b.$$

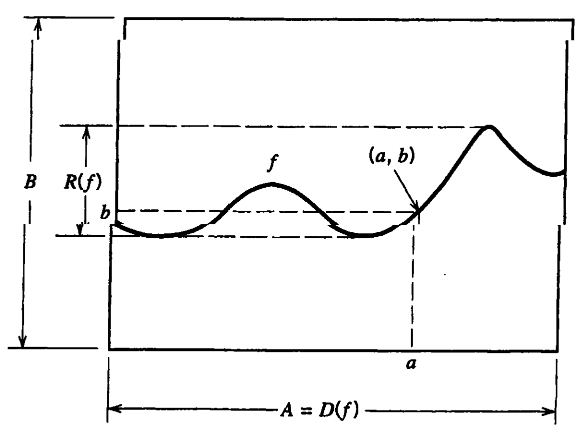

Figure 1.1.4 A function as a graph

If $b = f(a)$, we often refer to b as the **value** of f at a, or as the **image** of a under f.

Transformations and Machines

Aside from using graphs, we can visualize a function as a *transformation* of the set $D(f) = A$ into the set $R(f) \subseteq B$. In this phraseology, when $(a, b) \in f$, we think of f as taking the element a from A and "transforming" or "mapping" it into an element $b = f(a)$ in $R(f) \subseteq B$. We often draw a diagram, such as Figure 1.1.5, even when the sets A and B are not subsets of the plane.

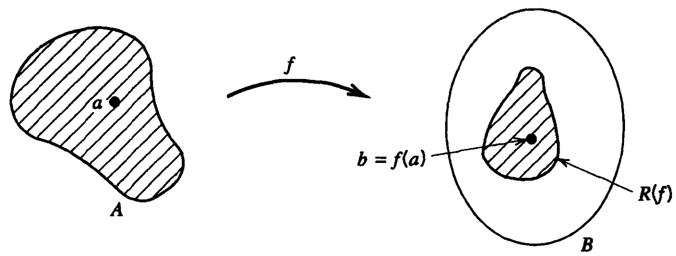

Figure 1.1.5 A function as a transformation

There is another way of visualizing a function: namely, as a *machine* that accepts elements of $D(f) = A$ as *inputs* and produces corresponding elements of $R(f) \subseteq B$ as *outputs*. If we take an element $x \in D(f)$ and put it into f, then out comes the corresponding value $f(x)$. If we put a different element $y \in D(f)$ into f, then out comes $f(y)$ which may or may not differ from $f(x)$. If we try to insert something that does not belong to $D(f)$ into f, we find that it is not accepted, for f can operate only on elements from $D(f)$. (See Figure 1.1.6.)

This last visualization makes clear the distinction between f and $f(x)$: the first is the machine itself, and the second is the output of the machine f when x is the input. Whereas no one is likely to confuse a meat grinder with ground meat, enough people have confused functions with their values that it is worth distinguishing between them notationally.

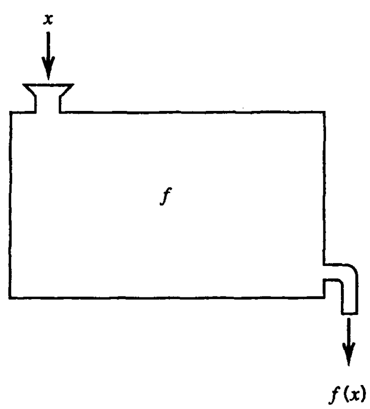

Figure 1.1.6 A function as a machine

Direct and Inverse Images

Let $f : A \to B$ be a function with domain $D(f) = A$ and range $R(f) \subseteq B$.

1.1.7 Definition If E is a subset of A, then the **direct image** of E under f is the subset $f(E)$ of B given by

$$f(E) := \{f(x) : x \in E\}.$$

If H is a subset of B, then the **inverse image** of H under f is the subset $f^{-1}(H)$ of A given by

$$f^{-1}(H) := \{x \in A : f(x) \in H\}.$$

Remark The notation $f^{-1}(H)$ used in this connection has its disadvantages. However, we will use it since it is the standard notation.

Thus, if we are given a set $E \subseteq A$, then a point $y_1 \in B$ is in the direct image $f(E)$ if and only if there exists at least one point $x_1 \in E$ such that $y_1 = f(x_1)$. Similarly, given a set $H \subseteq B$, then a point x_2 is in the inverse image $f^{-1}(H)$ if and only if $y_2 := f(x_2)$ belongs to H. (See Figure 1.1.7.)

1.1.8 Examples **(a)** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := x^2$. Then the direct image of the set $E := \{x : 0 \le x \le 2\}$ is the set $f(E) = \{y : 0 \le y \le 4\}$.

If $G := \{y : 0 \le y \le 4\}$, then the inverse image of G is the set $f^{-1}(G) = \{x : -2 \le x \le 2\}$. Thus, in this case, we see that $f^{-1}(f(E)) \neq E$.

On the other hand, we have $f\left(f^{-1}(G)\right) = G$. But if $H := \{y : -1 \le y \le 1\}$, then we have $f\left(f^{-1}(H)\right) = \{y : 0 \le y \le 1\} \neq H$.

A sketch of the graph of f may help to visualize these sets.

(b) Let $f : A \to B$, and let G, H be subsets of B. We will show that

$$f^{-1}(G \cap H) \subseteq f^{-1}(G) \cap f^{-1}(H).$$

For, if $x \in f^{-1}(G \cap H)$, then $f(x) \in G \cap H$, so that $f(x) \in G$ and $f(x) \in H$. But this implies that $x \in f^{-1}(G)$ and $x \in f^{-1}(H)$, whence $x \in f^{-1}(G) \cap f^{-1}(H)$. Thus the stated implication is proved. [The opposite inclusion is also true, so that we actually have set equality between these sets; see Exercise 13.] □

Further facts about direct and inverse images are given in the exercises.

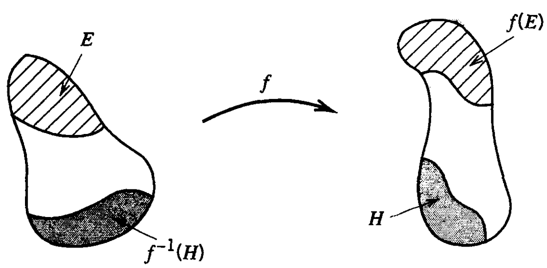

Figure 1.1.7 Direct and inverse images

Special Types of Functions

The following definitions identify some very important types of functions.

1.1.9 Definition Let $f : A \to B$ be a function from A to B.

(a) The function f is said to be **injective** (or to be **one-one**) if whenever $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. If f is an injective function, we also say that f is an **injection.**

(b) The function f is said to be **surjective** (or to map A **onto** B) if $f(A) = B$; that is, if the range $R(f) = B$. If f is a surjective function, we also say that f is a **surjection.**

(c) If f is both injective and surjective, then f is said to be **bijective.** If f is bijective, we also say that f is a **bijection.**

- In order to prove that a function f is injective, we must establish that:

  $$\text{for all } x_1, x_2 \text{ in } A, \text{ if } f(x_1) = f(x_2), \text{ then } x_1 = x_2.$$

  To do this we assume that $f(x_1) = f(x_2)$ and show that $x_1 = x_2$.
  [In other words, the graph of f satisfies the *first horizontal line test:* Every horizontal line $y = b$ with $b \in B$ intersects the graph f in *at most* one point.]
- To prove that a function f is surjective, we must show that for any $b \in B$ there exists at least one $x \in A$ such that $f(x) = b$.
  [In other words, the graph of f satisfies the *second horizontal line test:* Every horizontal line $y = b$ with $b \in B$ intersects the graph f in *at least* one point.]

1.1.10 Example Let $A := \{x \in \mathbb{R} : x \neq 1\}$ and define $f(x) := 2x/(x-1)$ for all $x \in A$. To show that f is injective, we take x_1 and x_2 in A and assume that $f(x_1) = f(x_2)$. Thus we have

$$\frac{2x_1}{x_1 - 1} = \frac{2x_2}{x_2 - 1},$$

which implies that $x_1(x_2 - 1) = x_2(x_1 - 1)$, and hence $x_1 = x_2$. Therefore f is injective.

To determine the range of f, we solve the equation $y = 2x/(x-1)$ for x in terms of y. We obtain $x = y/(y-2)$, which is meaningful for $y \neq 2$. Thus the range of f is the set $B := \{y \in \mathbb{R} : y \neq 2\}$. Thus, f is a bijection of A onto B. □

Inverse Functions

If f is a function from A into B, then f is a special subset of $A \times B$ (namely, one passing the *vertical line test.*) The set of ordered pairs in $B \times A$ obtained by interchanging the members of ordered pairs in f is not generally a function. (That is, the set f may not pass *both* of the *horizontal line tests.*) However, if f is a bijection, then this interchange does lead to a function, called the "inverse function" of f.

1.1.11 Definition If $f : A \to B$ is a bijection of A onto B, then

$$g := \{(b, a) \in B \times A : (a, b) \in f\}$$

is a function on B into A. This function is called the **inverse function** of f, and is denoted by f^{-1}. The function f^{-1} is also called the **inverse** of f.

We can also express the connection between f and its inverse f^{-1} by noting that $D(f) = R(f^{-1})$ and $R(f) = D(f^{-1})$ and that

$$b = f(a) \qquad \text{if and only if} \qquad a = f^{-1}(b).$$

For example, we saw in Example 1.1.10 that the function

$$f(x) := \frac{2x}{x-1}$$

is a bijection of $A := \{x \in \mathbb{R} : x \neq 1\}$ onto the set $B := \{y \in \mathbb{R} : y \neq 2\}$. The function inverse to f is given by

$$f^{-1}(y) := \frac{y}{y-2} \qquad \text{for} \quad y \in B.$$

Remark We introduced the notation $f^{-1}(H)$ in Definition 1.1.7. It makes sense even if f does not have an inverse function. However, if the inverse function f^{-1} does exist, then $f^{-1}(H)$ is the direct image of the set $H \subseteq B$ under f^{-1}.

Composition of Functions

It often happens that we want to "compose" two functions f, g by first finding $f(x)$ and then applying g to get $g(f(x))$; however, this is possible only when $f(x)$ belongs to the domain of g. In order to be able to do this for *all* $f(x)$, we must assume that the range of f is contained in the domain of g. (See Figure 1.1.8.)

1.1.12 Definition If $f : A \to B$ and $g : B \to C$, and if $R(f) \subseteq D(g) = B$, then the **composite function** $g \circ f$ (note the order!) is the function from A into C defined by

$$(g \circ f)(x) := g(f(x)) \qquad \text{for all} \quad x \in A.$$

1.1.13 Examples **(a)** The order of the composition must be carefully noted. For, let f and g be the functions whose values at $x \in \mathbb{R}$ are given by

$$f(x) := 2x \qquad \text{and} \qquad g(x) := 3x^2 - 1.$$

Since $D(g) = \mathbb{R}$ and $R(f) \subseteq \mathbb{R} = D(g)$, then the domain $D(g \circ f)$ is also equal to $\mathbb{R}$, and the composite function $g \circ f$ is given by

$$(g \circ f)(x) = 3(2x)^2 - 1 = 12x^2 - 1.$$

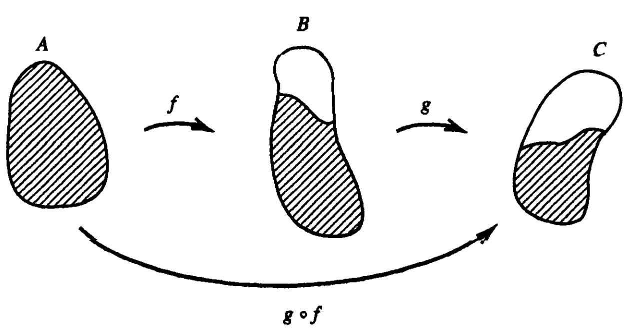

Figure 1.1.8 The composition of f and g

On the other hand, the domain of the composite function $f \circ g$ is also $\mathbb{R}$, but

$$(f \circ g)(x) = 2(3x^2 - 1) = 6x^2 - 2.$$

Thus, in this case, we have $g \circ f \neq f \circ g$.

(b) In considering $g \circ f$, some care must be exercised to be sure that the range of f is contained in the domain of g. For example, if

$$f(x) := 1 - x^2 \qquad \text{and} \qquad g(x) := \sqrt{x},$$

then, since $D(g) = \{x : x \geq 0\}$, the composite function $g \circ f$ is given by the formula

$$(g \circ f)(x) = \sqrt{1 - x^2}$$

only for $x \in D(f)$ that satisfy $f(x) \geq 0$; that is, for x satisfying $-1 \leq x \leq 1$.

We note that if we reverse the order, then the composition $f \circ g$ is given by the formula

$$(f \circ g)(x) = 1 - x,$$

but only for those x in the domain $D(g) = \{x : x \geq 0\}$. □

We now give the relationship between composite functions and inverse images. The proof is left as an instructive exercise.

1.1.14 Theorem *Let $f : A \to B$ and $g : B \to C$ be functions and let H be a subset of C. Then we have*

$$(g \circ f)^{-1}(H) = f^{-1}(g^{-1}(H)).$$

Note the *reversal* in the order of the functions.

Restrictions of Functions

If $f : A \to B$ is a function and if $A_1 \subset A$, we can define a function $f_1 : A_1 \to B$ by

$$f_1(x) := f(x) \qquad \text{for} \quad x \in A_1.$$

The function f_1 is called the **restriction of** f **to** A_1. Sometimes it is denoted by $f_1 = f|A_1$.

It may seem strange to the reader that one would ever choose to throw away a part of a function, but there are some good reasons for doing so. For example, if $f : \mathbb{R} \to \mathbb{R}$ is the **squaring function**:

$$f(x) := x^2 \qquad \text{for} \quad x \in \mathbb{R},$$

then f is not injective, so it cannot have an inverse function. However, if we restrict f to the set $A_1 := \{x : x \geq 0\}$, then the restriction $f|A_1$ is a bijection of A_1 onto A_1. Therefore, this restriction has an inverse function, which is the **positive square root function**. (Sketch a graph.)

Similarly, the trigonometric functions $S(x) := \sin x$ and $C(x) := \cos x$ are not injective on all of $\mathbb{R}$. However, by making suitable restrictions of these functions, one can obtain the **inverse sine** and the **inverse cosine** functions that the reader has undoubtedly already encountered.

Exercises

1. If A ar
2. Prove
3. Prove
(a) A
(b) A
4. The sy
A or B
(a) S
(b) S
5. For ea
(a) W
(b) D
6. Draw
(a) A
(b) A
7. Let A
$A \times B$
8. Let $f($
(a) D
(b) D
9. Let $g($
(a) Fi
(b) Fi
10. Let $f($
Show
Hence
E and
11. Let f
true th
12. Show
$f(E \cap$
13. Show
and f^{-}
14. Show
$\{y : -$
15. For a,
$y < 1]$
16. Give a
17. (a) S
t
(b) S
t
18. (a) S
f
(b) I

Exercises for Section 1.1

1. If A and B are sets, show that $A \subseteq B$ if and only if $A \cap B = A$.
2. Prove the second De Morgan Law [Theorem 1.1.4(b)].
3. Prove the Distributive Laws:
   (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
   (b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
4. The **symmetric difference** of two sets A and B is the set D of all elements that belong to either A or B but not both. Represent D with a diagram.
   (a) Show that $D = (A \backslash B) \cup (B \backslash A)$.
   (b) Show that D is also given by $D = (A \cup B) \backslash (A \cap B)$.
5. For each $n \in \mathbb{N}$, let $A_n = \{(n+1)k : k \in \mathbb{N}\}$.
   (a) What is $A_1 \cap A_2$?
   (b) Determine the sets $\bigcup\{A_n : n \in \mathbb{N}\}$ and $\bigcap\{A_n : n \in \mathbb{N}\}$.
6. Draw diagrams in the plane of the Cartesian products $A \times B$ for the given sets A and B.
   (a) $A = \{x \in \mathbb{R} : 1 \le x \le 2 \text{ or } 3 \le x \le 4\}$, $B = \{x \in \mathbb{R} : x = 1 \text{ or } x = 2\}$.
   (b) $A = \{1, 2, 3\}$, $B = \{x \in \mathbb{R} : 1 \le x \le 3\}$.
7. Let $A := B := \{x \in \mathbb{R} : -1 \le x \le 1\}$ and consider the subset $C := \{(x, y) : x^2 + y^2 = 1\}$ of $A \times B$. Is this set a function? Explain.
8. Let $f(x) := 1/x^2$, $x \neq 0$, $x \in \mathbb{R}$.
   (a) Determine the direct image $f(E)$ where $E := \{x \in \mathbb{R} : 1 \le x \le 2\}$.
   (b) Determine the inverse image $f^{-1}(G)$ where $G := \{x \in \mathbb{R} : 1 \le x \le 4\}$.
9. Let $g(x) := x^2$ and $f(x) := x + 2$ for $x \in \mathbb{R}$, and let h be the composite function $h := g \circ f$.
   (a) Find the direct image $h(E)$ of $E := \{x \in \mathbb{R} : 0 \le x \le 1\}$.
   (b) Find the inverse image $h^{-1}(G)$ of $G := \{x \in \mathbb{R} : 0 \le x \le 4\}$.
10. Let $f(x) := x^2$ for $x \in \mathbb{R}$, and let $E := \{x \in \mathbb{R} : -1 \le x \le 0\}$ and $F := \{x \in \mathbb{R} : 0 \le x \le 1\}$. Show that $E \cap F = \{0\}$ and $f(E \cap F) = \{0\}$, while $f(E) = f(F) = \{y \in \mathbb{R} : 0 \le y \le 1\}$. Hence $f(E \cap F)$ is a proper subset of $f(E) \cap f(F)$. What happens if 0 is deleted from the sets E and F?
11. Let f and E, F be as in Exercise 10. Find the sets $E \backslash F$ and $f(E) \backslash f(F)$ and show that it is *not* true that $f(E \backslash F) \subseteq f(E) \backslash f(F)$.
12. Show that if $f : A \to B$ and E, F are subsets of A, then $f(E \cup F) = f(E) \cup f(F)$ and $f(E \cap F) \subseteq f(E) \cap f(F)$.
13. Show that if $f : A \to B$ and G, H are subsets of B, then $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$ and $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$.
14. Show that the function f defined by $f(x) := x/\sqrt{x^2 + 1}$, $x \in \mathbb{R}$, is a bijection of $\mathbb{R}$ onto $\{y : -1 < y < 1\}$.
15. For $a, b \in \mathbb{R}$ with $a < b$, find an explicit bijection of $A := \{x : a < x < b\}$ onto $B := \{y : 0 < y < 1\}$.
16. Give an example of two functions f, g on $\mathbb{R}$ to $\mathbb{R}$ such that $f \neq g$, but such that $f \circ g = g \circ f$.
17. (a) Show that if $f : A \to B$ is injective and $E \subseteq A$, then $f^{-1}(f(E)) = E$. Give an example to show that equality need not hold if f is not injective.
    (b) Show that if $f : A \to B$ is surjective and $H \subseteq B$, then $f(f^{-1}(H)) = H$. Give an example to show that equality need not hold if f is not surjective.
18. (a) Suppose that f is an injection. Show that $f^{-1} \circ f(x) = x$ for all $x \in D(f)$ and that $f \circ f^{-1}(y) = y$ for all $y \in R(f)$.
    (b) If f is a bijection of A onto B, show that f^{-1} is a bijection of B onto A.