# Application of Linear Algebra to Differential Equations

## Segment 5: More Examples

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## OUTLINE

- Segment 1. Introduction; the equation $Y' = AY$
- Segment 2. The matrix exponential
- Segment 3. Spectral Mapping Theorem for matrix exponential
- Segment 4. Some easy examples
- **Segment 5. More examples**
- Segment 6. Complication: $A$ not diagonalizable
- Segment 7. An example with $A$ not diagonalizable

**References:** Section 8.3, Section 10.2

**Problems:** For Discussion May 1: page 328: 1, 2, 3, 4, 5   page 392: 1, 2, 4

Again we want to use the results of Segments 2 and 3:

**Theorem:** *If $A$ is an $n \times n$ matrix and $C$ is a vector in $\mathbb{R}^n$ or $\mathbb{C}^n$, then the function $Y(t) = e^{tA}C$ is the unique solution of the initial value problem:* $Y' = AY$ *and* $Y(0) = C$

and also:

**Theorem:**

*If $A$ is an $n \times n$ matrix and $v_1, v_2, \cdots, v_n$ is a basis for $\mathbb{C}^n$ consisting of eigenvectors for $A$ associated with the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$,*

*then the unique solution of the initial value problem: $Y' = AY$, $Y(0) = C$*

*is* $Y(t) = \alpha_1 e^{\lambda_1 t} v_1 + \alpha_2 e^{\lambda_2 t} v_2 + \cdots + \alpha_n e^{\lambda_n t} v_n$,

*where* $C = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$

**Example:**

Solve the initial value problem: $\begin{cases} y_1' = 2y_1 + y_2 \\ y_2' = -y_1 + 2y_2 \end{cases}$ and $\begin{cases} y_1(0) = 3 \\ y_2(0) = 0 \end{cases}$

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We can rewrite this as $Y' = FY$ and $Y(0) = R$ by choosing

$$Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad F = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \text{ and } R = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

As before, we will use Matlab to do the calculations.

**Example (cont'd):**

From the Matlab computations, the solution of $Y' = FY$, $Y(0) = R$ is

$$Y(t) = c_1 e^{\lambda_1 t} w_1 + c_2 e^{\lambda_2 t} w_2$$

where $c_1 = 2.1213$, $c_2 = 2.1213$

$$w_1 = \begin{pmatrix} 0.7071 \\ 0.7071i \end{pmatrix}, \quad w_2 = \begin{pmatrix} 0.7071 \\ -0.7071i \end{pmatrix},$$

and $\lambda_1 = 2 + i$, and $\lambda_2 = 2 - i$

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Since $\lambda_1$ and $\lambda_2$ are different, the eigenvalues of $F$ are distinct and $F$ is diagonalizable. In particular, this means we get the solution exactly as before:

$$Y(t) = c_1 e^{(2+i)t} w_1 + c_2 e^{(2-i)t} w_2 = e^{(2+i)t} \begin{pmatrix} 1.5 \\ 1.5i \end{pmatrix} + e^{(2-i)t} \begin{pmatrix} 1.5 \\ -1.5i \end{pmatrix}$$

**Example (cont'd):**

On the other hand, our IVP: $\begin{cases} y_1' = 2y_1 + y_2 \\ y_2' = -y_1 + 2y_2 \end{cases}$ and $\begin{cases} y_1(0) = 3 \\ y_2(0) = 0 \end{cases}$ is real

and our answer is complex(!): $\quad Y(t) = e^{(2+i)t} \begin{pmatrix} 1.5 \\ 1.5i \end{pmatrix} + e^{(2-i)t} \begin{pmatrix} 1.5 \\ -1.5i \end{pmatrix},$

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From calculus, we know $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$ for all real numbers $x$, and that the series converges absolutely.

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(BUT we will see it does NOT hold for $n \times n$ matrices for $n \geq 2$!!)

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If $a$ and $b$ are real numbers $e^{a+bi} = e^a e^{ib}$, and we understand $e^a$.

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Applying this to the complex exponential function in the Example, we see

$e^{(2+i)t} = e^{2t}e^{it} = e^{2t}(\cos(t) + i\sin(t))$ and $e^{(2-i)t} = e^{2t}(\cos(t) - i\sin(t))$ and

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$$Y(t) = e^{(2+i)t}\begin{pmatrix} 1.5 \\ 1.5i \end{pmatrix} + e^{(2-i)t}\begin{pmatrix} 1.5 \\ -1.5i \end{pmatrix}$$

$$= 1.5e^{2t}\left[\begin{pmatrix} (\cos(t) + i\sin(t)) \\ (\cos(t) + i\sin(t))i \end{pmatrix} + \begin{pmatrix} (\cos(t) - i\sin(t)) \\ -(\cos(t) - i\sin(t))i \end{pmatrix}\right]$$

$$= \begin{pmatrix} 3e^{2t}\cos(t) \\ -3e^{2t}\sin(t) \end{pmatrix}$$ or, $y_1(t) = 3e^{2t}\cos(t)$ and $y_2(t) = -3e^{2t}\sin(t)$

## Another Example:

Solve initial value problem: $y'' + 5y' + 6y = 0$ with $y(0) = 1$ and $y'(0) = -1$

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Since $y'' + 5y' + 6y = 0$, we see $y_2' = y'' = -5y' - 6y = -6y_1 - 5y_2$.

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The resulting IVP is: $\begin{cases} y_1' = y_2 \\ y_2' = -6y_1 - 5y_2 \end{cases}$ and $\begin{cases} y_1(0) = 1 \\ y_2(0) = -1 \end{cases}$

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We solve this system in the usual way, and interpret it as above!

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We can rewrite this as $Y' = GY$ and $Y(0) = S$ by choosing

$$Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad G = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \text{ and } S = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Solution of $Y' = GY$, $Y(0) = S$ is $Y(t) = d_1 e^{\lambda_1 t} x_1 + d_2 e^{\lambda_2 t} x_2$

where $d_1 = 4.4721$, $d_2 = 3.1623$, $\lambda_1 = -2$, $\lambda_2 = -3$,

$$x_1 = \begin{pmatrix} 0.4472 \\ -0.8944 \end{pmatrix}, \text{ and } x_2 = \begin{pmatrix} -0.3162 \\ 0.9487 \end{pmatrix}$$

**Another Example:**

In other words, the function

$$Y(t) = d_1 e^{-2t} x_1 + d_2 e^{-3t} x_2 \;=\; e^{-2t} \begin{pmatrix} 2 \\ -4 \end{pmatrix} + e^{-3t} \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

is the solution of the IVP: $\begin{cases} y_1' = y_2 \\ y_2' = -6y_1 - 5y_2 \end{cases}$ and $\begin{cases} y_1(0) = 1 \\ y_2(0) = -1 \end{cases}$

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In particular, relating this to the original IVP, this means

$$y(t) = y_1(t) = 2e^{-2t} - e^{-3t}$$

is the solution of the initial value problem:

$$y'' + 5y' + 6y = 0 \text{ with } y(0) = 1 \text{ and } y'(0) = -1.$$

This is the end of the Fifth Segment.

In the next segment, we tackle initial value problems

for which the coefficient matrix is non-diagonalizable.