Application of Linear Algebra to Differential Equations

Segment 4: Some Easy Examples

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OUTLINE

- Segment 1. Introduction; the equation $Y' = AY$
- Segment 2. The matrix exponential
- Segment 3. Spectral Mapping Theorem for matrix exponential
- **Segment 4. Some easy examples**
- Segment 5. More examples
- Segment 6. Complication: A not diagonalizable
- Segment 7. An example with A not diagonalizable

References: Section 8.3, Section 10.2

Problems: For Discussion May 1: page 328: 1, 2, 3, 4, 5 page 392: 1, 2, 4

In Segment 2, we proved the following result:

Theorem: *If A is an $n \times n$ matrix and C is a vector in $\mathbb{R}^n$ or $\mathbb{C}^n$, then the function $Y(t) = e^{tA}C$ is the unique solution of the initial value problem:* $\quad Y' = AY$ *and* $Y(0) = C$

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of the initial value problem: $Y' = AY$ *and* $Y(0) = C$

In the last Segment, we described a way to calculate the function $Y(t)$ in the case A is diagonalizable.

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From Segment 3:

Theorem:

If A is an $n \times n$ matrix and $v_1, v_2, \cdots, v_n$ is a basis for $\mathbb{C}^n$ consisting of

eigenvectors for A associated with the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$,

then the unique solution of the initial value problem: $Y' = AY$, $Y(0) = C$

is $Y(t) = \alpha_1 e^{\lambda_1 t} v_1 + \alpha_2 e^{\lambda_2 t} v_2 + \cdots + \alpha_n e^{\lambda_n t} v_n$,

where $C = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$

Example:

Solve the initial value problem: $\begin{cases} y_1' = 12y_1 + 10y_2 \\ y_2' = -15y_1 - 13y_2 \end{cases}$ and $\begin{cases} y_1(0) = 3 \\ y_2(0) = 1 \end{cases}$

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This initial value problem is

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 12 & 10 \\ -15 & -13 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

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Letting $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ $\quad A = \begin{pmatrix} 12 & 10 \\ -15 & -13 \end{pmatrix}$ and $C = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

we can rewrite the initial value problem above as $Y' = AY$ and $Y(0) = C$

which has the form as stated in the Theorem describing the solution.

Example (cont'd):

According to Theorem, to solve initial value problem $Y' = AY, \; Y(0) = C$

$$\text{where } Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad A = \begin{pmatrix} 12 & 10 \\ -15 & -13 \end{pmatrix} \text{ and } C = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

we need to decide if A is diagonalizable, and if it is, write C as a linear combination of an appropriate eigenvector basis for $\mathbb{C}^2$, then write the solution.

We will use Matlab to do the relevant computations.

Example (cont'd):

From the Matlab computations, the solution of $Y' = AY$, $Y(0) = C$ is

$Y(t) = a_1 e^{\lambda_1 t} v_1 + a_2 e^{\lambda_2 t} v_2$ where $a_1 = 15.5563$, $a_2 = 14.4222$,

$$v_1 = \begin{pmatrix} 0.7071 \\ -0.7071 \end{pmatrix}, \; v_2 = \begin{pmatrix} -0.5547 \\ 0.8321 \end{pmatrix}, \qquad \lambda_1 = 2 \text{ and } \lambda_2 = -3$$

so $Y(t) = e^{2t} \begin{pmatrix} 11 \\ -11 \end{pmatrix} + e^{-3t} \begin{pmatrix} -8 \\ 12 \end{pmatrix}$,

which means $y_1(t) = 11e^{2t} - 8e^{-3t}$ and $y_2(t) = -11e^{2t} + 12e^{-3t}$

Example:

Our goal was to solve the IVP: $\begin{cases} y_1' &= \quad 12y_1 \;+\; 10y_2 \\ y_2' &= -15y_1 \;-\; 13y_2 \end{cases}$ and $\begin{cases} y_1(0) = 3 \\ y_2(0) = 1 \end{cases}$

and our solution was $y_1(t) = 11e^{2t} - 8e^{-3t}$ and $y_2(t) = -11e^{2t} + 12e^{-3t}$

Check!! We should check this answer:

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Our goal was to solve the IVP: $\begin{cases} y_1' &= 12y_1 + 10y_2 \\ y_2' &= -15y_1 - 13y_2 \end{cases}$ and $\begin{cases} y_1(0) = 3 \\ y_2(0) = 1 \end{cases}$

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First, $y_1(0) = 11e^{2\cdot 0} - 8e^{-3\cdot 0} = 11 - 8 = 3$ and $y_2(0) = -11 + 12 = 1$

so the initial conditions are satisfied.

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Second, $12y_1(t) + 10y_2(t) = 132e^{2t} - 96e^{-3t} - 110e^{2t} + 120e^{-3t} = 22e^{2t} + 24e^{-3t}$

and $y_1'(t) = 22e^{2t} + 24e^{-3t}$, so $y_1' = 12y_1 + 10y_2$, as we wished.

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Also, $-15y_1 - 13y_2 = -165e^{2t} + 120e^{-3t} + 143e^{2t} - 156e^{-3t} = -22e^{2t} - 36e^{-3t}$

and $y_2'(t) = -22e^{2t} - 36e^{-3t}$, so $y_2' = -15y_1 - 13y_2$, as we wished.

Another Example:

Solve the IVP: $\begin{cases} y_1' = \phantom{4y_1 +{}} 4y_2 + 2y_3 \\ y_2' = 4y_1 \phantom{{}+ 2y_2} + 2y_3 \\ y_3' = 2y_1 + 2y_2 - 3y_3 \end{cases}$ and $\begin{cases} y_1(0) = 1 \\ y_2(0) = -3 \\ y_3(0) = 2 \end{cases}$

We can rewrite this as $Y' = BY$ and $Y(0) = Q$ by choosing

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 4 & 2 \\ 4 & 0 & 2 \\ 2 & 2 & -3 \end{pmatrix} \text{ and } Q = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$$

We notice that B is Hermitian, so it is diagonalizable

and the Theorem we want to use holds for this problem.

Again, we will use Matlab to do the calculations.

Example (cont'd):

From the Matlab computations, the solution of $Y' = BY$, $Y(0) = Q$ is

$$Y(t) = b_1 e^{\lambda_1 t} u_1 + b_2 e^{\lambda_2 t} u_2 + b_3 e^{\lambda_3 t} u_3$$

where $b_1 = 2.1125$, $b_2 = -3.0154$, $b_3 = 0.6667$,

$$u_1 = \begin{pmatrix} -0.2939 \\ -0.1758 \\ 0.9395 \end{pmatrix}, \; u_2 = \begin{pmatrix} -0.6849 \\ 0.7243 \\ -0.0788 \end{pmatrix}, \; u_3 = \begin{pmatrix} -0.6667 \\ -0.6667 \\ -0.3333 \end{pmatrix},$$

and $\lambda_1 = -4$, $\lambda_2 = -4$, and $\lambda_3 = 5$

Since $\lambda_1 = \lambda_2 = -4$, the expression for Y simplifies:

$$Y(t) = b_1 e^{-4t} u_1 + b_2 e^{-4t} u_2 + b_3 e^{5t} u_3 = (b_1 u_1 + b_2 u_2) e^{-4t} + b_3 e^{5t} u_3$$

Example (cont'd):

That is, $Y(t) = (b_1 u_1 + b_2 u_2)e^{-4t} + b_3 e^{5t} u_3$

where $b_1 = 2.1125,\ b_2 = -3.0154,\ b_3 = 0.6667,$

$$u_1 = \begin{pmatrix} -0.2939 \\ -0.1758 \\ 0.9395 \end{pmatrix},\ u_2 = \begin{pmatrix} -0.6849 \\ 0.7243 \\ -0.0788 \end{pmatrix},\ \text{and}\ \ u_3 = \begin{pmatrix} -0.6667 \\ -0.6667 \\ -0.3333 \end{pmatrix}$$

Combining the calculations, we see

$$y_1(t) = \frac{1}{9}\left(13e^{-4t} - 4e^{5t}\right)$$

$$y_2(t) = \frac{1}{9}\left(-23e^{-4t} - 4e^{5t}\right)$$

$$y_3(t) = \frac{1}{9}\left(20e^{-4t} - 2e^{5t}\right)$$

and some tedious calculations verify that this solves the IVP.

This is the end of the Fourth Segment.

In the next segment, we do some slightly different kinds of examples that use the same basic theorem for solution.