Application of Linear Algebra to Differential Equations

Segment 3: Spectral Mapping Theorem for the Matrix Exponential

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OUTLINE

- Segment 1. Introduction; the equation $Y' = AY$
- Segment 2. The matrix exponential
- **Segment 3. Spectral Mapping Thm for matrix exponential**
- Segment 4. Some easy examples
- Segment 5. More examples
- Segment 6. Complication: A not diagonalizable
- Segment 7. An example with A not diagonalizable

References: Section 8.3, Section 10.2

Problems: For Discussion May 1: page 328: 1, 2, 3, 4, 5 page 392: 1, 2, 4

Definition (*Matrix Exponential Function*):

If A is $n \times n$ matrix, the matrix exponential function e^{tA} is defined by series

$$e^{tA} = I + tA + \frac{(tA)^2}{2!} + \frac{(tA)^3}{3!} + \frac{(tA)^4}{4!} + \cdots$$

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In this segment, we wish to use the Spectral Mapping Theorem to be able to effectively compute $e^{tA}v$ for any number t, any matrix A, and any vector v.

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Indeed, suppose

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Then $e^{tD} = I + tD + \dfrac{t^2}{2!}D^2 + \cdots$

For $D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}$ we have

$$D^2 = \begin{pmatrix} d_1^2 & 0 & \cdots & 0 \\ 0 & d_2^2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^2 \end{pmatrix} \quad \text{and} \quad D^3 = \begin{pmatrix} d_1^3 & 0 & \cdots & 0 \\ 0 & d_2^3 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^3 \end{pmatrix} \quad \text{etc.}$$

For $D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}$ this means

$$e^{tD} = I + tD + \cdots = \begin{pmatrix} 1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & 1 \end{pmatrix} + t \begin{pmatrix} d_1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & d_n \end{pmatrix} + \frac{t^2}{2!} \begin{pmatrix} d_1^2 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & d_n^2 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & 1 \end{pmatrix} + \begin{pmatrix} td_1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & td_n \end{pmatrix} + \begin{pmatrix} \frac{(td_1)^2}{2!} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & \frac{(td_n)^2}{2!} \end{pmatrix} + \cdots$$

For D diagonal with diagonal entries d_j, this means

$$e^{tD} = I + tD + \frac{(tD)^2}{2!} + \cdots =$$

$$= \begin{pmatrix} 1 + td_1 + \frac{(td_1)^2}{2!} + \cdots & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & 1 + td_n + \frac{(td_n)^2}{2!} + \cdots \end{pmatrix}$$

$$= \begin{pmatrix} e^{td_1} & 0 & \cdots & 0 \\ 0 & e^{td_2} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e^{td_n} \end{pmatrix}$$

However, most matrices are much harder to exponentiate!

For example if $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$, then $A^2 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}$

$$A^3 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 7 \\ 0 & 8 \end{pmatrix}, \quad A^4 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 7 \\ 0 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 15 \\ 0 & 16 \end{pmatrix}$$

and

$$e^{tA} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} + \frac{t^2}{2!} \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix} + \frac{t^3}{3!} \begin{pmatrix} 1 & 7 \\ 0 & 8 \end{pmatrix} + \cdots = \begin{pmatrix} e^t & ?? \\ 0 & e^{2t} \end{pmatrix}$$

Theorem:

If A is an $n \times n$ matrix with eigenvector v with eigenvalue λ,

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$$\begin{aligned} e^{tA}v &= \left(I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots\right)v \\ &= Iv + tAv + \frac{t^2}{2!}A^2v + \frac{t^3}{3!}A^3v + \cdots \end{aligned}$$

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&= Iv + tAv + \frac{t^2}{2!}A^2v + \frac{t^3}{3!}A^3v + \cdots \\
&= v + \lambda t v + \frac{(\lambda t)^2}{2!}v + \frac{(\lambda t)^3}{3!}v + \cdots
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Corollary:

Let A be an $n \times n$ matrix with eigenvectors $v_1, v_2, \cdots, v_k$ corresponding to the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$.

If $C = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k$

then $e^{tA} C = \alpha_1 e^{\lambda_1 t} v_1 + \alpha_2 e^{\lambda_2 t} v_2 + \cdots + \alpha_k e^{\lambda_k t} v_k$

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In particular, if A is diagonalizable, there is a basis for $\mathbb{C}^n$ consisting of eigenvectors of A and this corollary gives the solution of *every* initial value problem for the differential equation $Y' = AY$.

This is the end of the Third Segment.

In the next segment, we will begin with this result and use it to solve some initial value problems.