Application of Linear Algebra to Differential Equations

Segment 2: The Matrix Exponential

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Math 35300, April 26, 2014

OUTLINE

- Segment 1. Introduction; the equation $Y' = AY$
- **Segment 2. The matrix exponential**
- Segment 3. Spectral Mapping Theorem for the matrix exponential
- Segment 4. Some easy examples
- Segment 5. More examples
- Segment 6. Complication: A not diagonalizable
- Segment 7. An example with A not diagonalizable

References: Section 8.3, Section 10.2

Problems: For Discussion May 1: page 328: 1, 2, 3, 4, 5   page 392: 1, 2, 4

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Definition (*Matrix Exponential*):

If B is an $n \times n$ matrix, the matrix e^B is defined by the series

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We need to show that this definition makes sense, that is, that the series converges, and learn about the properties of the matrix exponential.

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- The function $Y(t) = e^{tA}C$ satisfies the initial condition $Y(0) = C$
- The function $Y(t) = e^{tA}C$ is the only solution of the initial value problem:

  $Y' = AY$ and $Y(0) = C$

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which converges to $e^{|t|\|A\|}$

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converges, so the series for e^{tA} is absolutely convergent, therefore, convergent. Moreover, the estimate above shows that $\left\|e^{tA}\right\|$ is no more than $e^{|t|\|A\|}$ and the series converges absolutely and uniformly for $-M \le t \le M$ for each positive number M. Thus, the series can be differentiated term by term and

$$\begin{aligned} \frac{d}{dt}e^{tA} &= 0 + A + \frac{2tA^2}{2!} + \frac{3t^2A^3}{3!} + \frac{4t^3A^4}{4!} + \cdots \\ &= A\left(I + \frac{tA}{1!} + \frac{t^2A^2}{2!} + \frac{t^3A^3}{3!} + \cdots\right) = Ae^{tA} \end{aligned}$$

so $Y'(t) = Ae^{tA}C = AY(t)$ and $Y(0) = e^0C = C$. ∎

This is the end of the Second Segment.

In the next segment, we will see how to use the Spectral Mapping Theorem to avoid the infinite series in this segment.

The goal is to be able to calculate $e^{tA}v$ for any number t, any matrix A, and any vector v.