Professor Carl Cowen **Math 44500** **Spring 11**

‘A’ LIST PROBLEMS

Solutions to problems from this list may be handed in at any time before 5:00pm on May 6. The problems will be read and either accepted as correct or returned for rewriting and resubmission. Only one of these problems will be counted for credit but this problem will be worth the same number of points as two regular homework assignments. In order to receive an ‘A’ or ‘A+’ for the course, you must have one of the *‘A’ List* problems accepted as correct.

1. (Construction of $\mathbb{R}$)
Peano (1889) and Dedekind (1888) gave a careful construction of the integers from the axioms for set theory. In an ‘A’ List problem for Math 44400, a construction of the rational numbers $\mathbb{Q}$ from the integers was outlined. We will take the rational numbers, $\mathbb{Q}$, and their properties as given. In particular, we assume as given the usual operations and the usual notation for the rational numbers as quotients of integers, as modified by the convention that two different quotients of integers, for example $1/3$ and $4/12$, are considered “=”. The goal of this problem is, using the rational numbers and their properties, to construct the set $\mathbb{R}$, define the operations of ‘addition’ and ‘multiplication’ for elements of $\mathbb{R}$, define the set $\mathcal{P}$ of ‘positive’ elements of $\mathbb{R}$, and prove that $\mathbb{R}$ with these operations and the distinguished subset $\mathcal{P}$ is an ordered field which we can recognize as being the (usual) real numbers.

An *equivalence relation* on a set $\mathcal{X}$ is a binary relation $\sim$ that satisfies (i) for every x in $\mathcal{X}$, $x \sim x$ (reflexivity), (ii) for x and y in $\mathcal{X}$, $x \sim y$ implies $y \sim x$ (symmetry), and (iii) for x, y, and z in $\mathcal{X}$, $x \sim y$ and $y \sim z$ implies $x \sim z$ (transitivity). A equivalence relation on a set can be used to define *equivalence classes*: For x in $\mathcal{X}$, the equivalence class of x, denoted $[x]$, is the subset of $\mathcal{X}$

$$[x] = \{y \in \mathcal{X} : y \sim x\}$$

From the properties of an equivalence relation, we see that $[x] = [y]$ if and only if $x \sim y$. An equivalence relation therefore allows us to break up a set into disjoint pieces, the equivalence classes. An easy example is modular arithmetic: we say integers m and n are equivalent modulo 2 if $m - n$ is divisible by 2. This equivalence relation breaks the integers into two disjoint subsets usually called the ‘even integers’ (the equivalence class of 2) and the ‘odd integers’ (the equivalence class of 1).

The Construction:

Let $\mathcal{Y}$ be the set of Cauchy sequences of rational numbers

$$\mathcal{Y} = \{(q_1, q_2, q_3, \cdots) : q_j \in \mathbb{Q} \text{ for each } j \text{ and the sequence } (q_j) \text{ is Cauchy }\}$$

where we say a sequence (q_j) is Cauchy if for each rational number $p > 0$, there is N in $\mathbb{N}$ so that $|q_j - q_k| < p$ for all $j > N$ and $k > N$.

We define a binary relation $\sim$ on $\mathcal{Y}$ by the following: for Cauchy sequences of rational numbers (p_k) and (q_j), we write $(p_k) \sim (q_j)$ when the sequence $p_1, q_1, p_2, q_2, p_3, q_3, \cdots$ is also Cauchy. In addition, we identify the rational number q with the equivalence class of the Cauchy sequence $q, q, q, q, \cdots$ and we let $\underline{q} = [(q, q, q, \cdots)]$. The set $\mathbb{R}$, of *real numbers*, is the set of equivalence classes of the sequences in $\mathcal{Y}$ with the operations defined below.

1. (Continued)

(a) Show that $\sim$, defined above, is an equivalence relation on $\mathcal{Y}$.

(b) Show that if p and q are distinct rational numbers, then the equivalence classes $\underline{p}$ and $\underline{q}$ are different. This shows that the map from $\mathbb{Q}$ into $\mathbb{R}$ given by $q \mapsto \underline{q}$ is injective. Of course, after we are finished with this problem and again think of the real numbers as a 'natural' object without any mysteries, this injective map will be considered the identification of $\mathbb{Q}$ as a *subset* of $\mathbb{R}$.

(c) Suppose (p_k) and (q_j) are in $\mathcal{Y}$ with $(p_k) \sim (q_j)$. Show that $(-p_k)$ and $(-q_j)$ are in $\mathcal{Y}$ and that $(-p_k) \sim (-q_j)$. This means that letting $\ominus[(q_j)]$ be $[(-q_j)]$ defines a real number and cannot cause confusion; we say "$\ominus[(q_j)] = [(-q_j)]$ is well-defined."

(d) Suppose (p_k), (p'_k), (q_j), and (q'_j) are in $\mathcal{Y}$ with $(p_k) \sim (p'_k)$ and $(q_j) \sim (q'_j)$. Show that $(p_k + q_k)$ and $(p'_k + q'_k)$ are in $\mathcal{Y}$ and $(p_k + q_k) \sim (p'_k + q'_k)$. Conclude that, for (p_k) and (q_k) in $\mathcal{Y}$, letting $[(p_k)] \oplus [(q_k)] = [(p_k + q_k)]$ is well-defined.

(e) Show that if p, q, and r are rational numbers with $p + q = r$, then as real numbers, $\underline{p} \oplus \underline{q} = \underline{r}$.

(f) Suppose that (p_k), (q_k), and (r_k) are Cauchy sequences of rational numbers in $\mathcal{Y}$ so that $\sigma = [(p_k)]$, $\theta = [(q_k)]$, and $\rho = [(r_k)]$ are real numbers. Prove that $\sigma \oplus \theta = \theta \oplus \sigma$ and that $(\sigma \oplus \theta) \oplus \rho = \sigma \oplus (\theta \oplus \rho)$, that is, that real number addition is commutative and associative.

(g) Show that if $\sigma = [(p_k)]$ is a real number, then $\sigma \oplus \underline{0} = \underline{0} \oplus \sigma = \sigma$, which means that $\underline{0}$ is the additive identity for the real numbers.

(h) Show that if $\sigma = [(p_k)]$ is a real number and $\ominus\sigma = [(-p_k)]$ as above, then $\sigma \oplus (\ominus\sigma) = (\ominus\sigma) \oplus \sigma = \underline{0}$, which means that $\ominus\sigma$ defined as above is the additive inverse for the real number σ.

(i) Suppose (p_k), (p'_k), (q_j), and (q'_j) are in $\mathcal{Y}$ with $(p_k) \sim (p'_k)$ and $(q_j) \sim (q'_j)$. Show that $(p_k q_k)$ and $(p'_k q'_k)$ are in $\mathcal{Y}$ and $(p_k q_k) \sim (p'_k q'_k)$. Conclude that, for (p_k) and (q_k) in $\mathcal{Y}$, letting $[(p_k)] \odot [(q_k)] = [(p_k q_k)]$ is well-defined.

(j) Show that if p, q, and r are rational numbers with $pq = r$, then as real numbers, $\underline{p} \odot \underline{q} = \underline{r}$.

(k) Suppose that (p_k), (q_k), and (r_k) are Cauchy sequences of rational numbers in $\mathcal{Y}$ so that $\sigma = [(p_k)]$, $\theta = [(q_k)]$, and $\rho = [(r_k)]$ are real numbers. Prove that $\sigma \odot \theta = \theta \odot \sigma$, that $(\sigma \odot \theta) \odot \rho = \sigma \odot (\theta \odot \rho)$, and that $\sigma \odot (\theta \oplus \rho) = \sigma \odot \theta \oplus \sigma \odot \rho$, that is, that real number multiplication is commutative and associative and that multiplication distributes over addition.

(l) Show that if $\sigma = [(p_k)]$ is a real number, then $\sigma \odot \underline{1} = \underline{1} \odot \sigma = \sigma$, which means that $\underline{1}$ is the multiplicative identity for the real numbers.

(m) Suppose (p_k) is a Cauchy sequence of non-zero rational numbers. Find a condition on the sequence that guarantees that the sequence (r_k), where $r_k = 1/p_k$ for each positive integer k, is also a Cauchy sequence. (Of course, you must express your condition in language that only uses the rational numbers!)

(n) Show that a Cauchy sequence (q_k) with $[(q_k)] = \underline{0}$ does not satisfy the condition you found in part (m), but that if (q_k) is a Cauchy sequence of rational numbers with $[(q_k)] \neq \underline{0}$, then there is a Cauchy sequence (p_k) which does satisfy your condition and $[(q_k)] = [(p_k)]$.

1. (Continued)

(o) Let (p_k) and (p'_k) be Cauchy sequences of rational numbers that satisfy the condition you found in part (m) and suppose $[(p_k)] = [(p'_k)]$. Show that if (r_k) and (r'_k) are the sequences for which $r_k = 1/p_k$ and $r'_k = 1/p'_k$ for each positive integer k, then $[(r_k)] = [(r'_k)]$. This, together with parts (m) and (n), means that for $[(q_k)] \neq \underline{0}$, letting $(1/[(q_k)]) = [(1/p_k)]$, where (p_k) is a Cauchy sequence of rational numbers satisfying your condition in part (m) and $[(q_k)] = [(p_k)]$, is well defined.

(p) Suppose $\sigma \neq \underline{0}$ and $(1/\sigma)$ is defined as in part (o). Show that $\sigma \odot (1/\sigma) = \underline{1}$, so that $(1/\sigma)$ is the multiplicative inverse of σ.

Note! In parts (a) through (p), you have shown that with these definitions, the set of equivalence classes of rational numbers that we are tentatively calling the set of *real numbers* is a field satisfying the axioms (A1), (A2), (A3), (A4), (M1), (M2), (M3), (M4), and (D) on page 23 of the text.

(q) Choose a set $\mathcal{P}$ of real numbers that contains $\underline{1}$ and deserves to be called the *positive real numbers*. Prove that your choice for $\mathcal{P}$ satisfies the order axioms (O1), (O2), and (O3) (or **(i), (ii), and (iii)**) on page 25 of the text. (Of course, you may use these order properties for the rational numbers in your work.)

The ordering of $\mathbb{R}$! Of course we order the real numbers using $\mathcal{P}$ by writing $\sigma > \rho$ if $\sigma \oplus (\ominus\rho)$ is in $\mathcal{P}$, and this makes $\mathbb{R}$ into an ordered field. Similarly, we define the absolute value $|\cdot|$ by $|\sigma| = \sigma$ if σ is in $\mathcal{P}$, $|\sigma| = \ominus\sigma$ if σ is in $-\mathcal{P}$, and $|\sigma| = \underline{0}$ if $\sigma = \underline{0}$. Following the ideas in Chapter 2 of the text, we define distance, neighborhoods, upper bounds, suprema, etc., and prove the theorems that depend on the order ideas and their extensions.

(r) Prove *The Completeness Property of* $\mathbb{R}$ (page 37 of the text) which, you will recall was an *assumption* about the real numbers as we used them in Math 44400. Your proof, in connection with the construction of $\mathbb{R}$ completed above, justifies this assumption about the real numbers. (Note that your proof of the supremum existing in $\mathbb{R}$ will involve your construction of a Cauchy sequence of rational numbers, or at least a proof of the existence of such a sequence, because the rationals are *NOT* complete and such equivalence classes are what we have created as "the real numbers".)

2. Let $\mathcal{F}$ be the family of anchored continuous real valued functions

$$\mathcal{F} = \{f : f \text{ is continuous on } [0,1] \text{ and } f(0) = f(1) = 0\}$$

If f is a function in $\mathcal{F}$ and ℓ is a number, ℓ is called a *flat of f* if there is a number a with $0 \leq a \leq a + \ell \leq 1$ and such that $f(a) = f(a + \ell)$. We call ℓ a *ubiquitous flat* if ℓ is a flat of every f in $\mathcal{F}$. Clearly $\ell = 0$ and $\ell = 1$ are ubiquitous flats. In addition, the reading problem from the recent Math 44400 Final Exam showed (in an equivalent form) that $\ell = 1/2$ is a ubiquitous flat. Find all ubiquitous flats (and prove your answer), that is, for each number r with $0 \leq r \leq 1$ decide if r is or is not a ubiquitous flat and give a proof or an example to justify your claim.

3. This exercise develops the Cantor set and the Cantor function. More properly, this is the *Cantor middle-thirds set* because one description of this set involves removal of the 'middle thirds' of intervals. Sets that are topologically the same can be achieved by removing middle fourths or middle fifths, or even middle portions of varying lengths, but doing so yields different sets.

In a discussion of decimal expansions, we could just as easily have used a positive integer (the *base*) different than 10. Most often encountered are the *binary* expansions in which the base is 2 and the digits used are 0 and 1. In this exercise, we will use *ternary* expansions, that is, using base 3 and the digits 0, 1, and 2. If r is a rational number of the form $p/3^k$ where $p < 3^k$, we write

$$\frac{p}{3^k} = (.d_1 d_2 d_3 \cdots d_k)_3 \quad \text{when} \quad \frac{p}{3^k} = \frac{d_1}{3^1} + \frac{d_2}{3^2} + \cdots + \frac{d_k}{3^k} \quad \text{for} \quad d_j \in \{0,1,2\}, j = 1, \cdots, k$$

This is similar to writing $3/8 = 375/10^3 = .375$ or $3/8 = 0/2 + 1/2^2 + 1/2^3 = (.011)_2$. Just as in decimal expansions, when the number is irrational or a rational that is *not* a fraction with a denominator 10^k for some integer k, we write infinite decimals, we can write finite or infinite ternary expansions for any real number. For example, $66/81 = 2/3 + 1/9 + 1/81 = (.2101)_3$ and $1/4 = (.0202020202\cdots)_3$. Notice that $1/3 = (.1000\cdots)_3 = (.02222\cdots)_3$ and indeed, we also have $66/81 = (.21002222\cdots)_3$. So we see that the ternary expansion of $1/4$ is unique, but every rational number of the form $p/3^k$ has *two* ternary expansions, one finite and one infinite.

Definition The *Cantor set*, $\mathcal{C}$, is the set

$$\mathcal{C} = \{x \in [0,1] : x \text{ has a ternary expansion whose digits consist } \textit{only} \text{ of 0's and 2's}\}$$

Thus, $0 = (.000\cdots)_3$, $1 = (.222\cdots)_3$, $2/3 = (.200\cdots)_3$, $1/3 = (.0222\cdots)_3$ and $1/4$ are all in $\mathcal{C}$, but $66/81 = (.210100\cdots)_3 = (.210022\cdots)_3$, $1/2 = (.1111\cdots)_3$, and $1/\sqrt{2} = (.2010021\cdots)_3$ are not.

(a) Show that the Cantor set $\mathcal{C}$ is a closed set.

(b) Suppose x is *not* in the Cantor set. Let $x_\ell = \sup\{y \in \mathcal{C} : y < x\}$ and let $x_u = \inf\{y \in \mathcal{C} : y > x\}$. Show that x_ℓ and x_u are in $\mathcal{C}$. Since x is *not* in the Cantor set, the ternary expansion of x, say $x = (.d_1 d_2 d_3 d_4 \cdots)_3$, has a 1 in it that cannot be eliminated by using a different ternary expansion for x and we suppose d_k is the first digit in the expansion of x that is 1. What are the ternary expansions of x_ℓ and x_u that consist only of 0's and 2's?

(c) Show that the set $\{y : y \notin \mathcal{C}\}$ is an open set and that it is dense in $[0,1]$, that is, if x is in $[0,1]$, every neighborhood of x contains a point of $[0,1]$ that is not in $\mathcal{C}$.

Definition The *Cantor function*, φ, is the function mapping $[0,1]$ into $[0,1]$ defined as follows: If x is in the Cantor set and $x = (.d_1 d_2 d_3 d_4 \cdots)_3$ where each d_j is either 0 or 2, then define $\varphi(x) = (.b_1 b_2 b_3 b_4 \cdots)_2$ where $b_j = d_j/2$. If x is not in the Cantor set, let $\varphi(x) = \varphi(x_\ell)$.

(d) Using your answer to part (b) above, show that if x is not in the Cantor set, then $\varphi(x) = \varphi(x_u)$ also.

(e) Show that the Cantor function, φ, is continuous and increasing on $[0,1]$.

(Problem continued on next page!)

3. (Continued)

(f) Show that the Cantor function maps $[0,1]$ onto $[0,1]$ and use this to show that the Cantor set, $\mathcal{C}$, is an uncountable set.

(g) Show that the Cantor function φ is Riemann integrable and find $\int_0^1 \varphi(t)\,dt$.

(h) At which points, x, of $[0,1]$ is φ differentiable? At these points, find $\varphi'(x)$.

4. Statement: *Let $a < b$ be given. For every countable set D with $D \subset [a,b]$, there is a function $f : [a,b] \to \mathbb{R}$ such that $D = \{x \in [a,b] : f$ is discontinuous at $x\}$.*

Either prove that the Statement is true or, if not, find a countable set D with $D \subset [0,1]$ that cannot be the set of discontinuities of any function defined on $[0,1]$.

5. For each positive integer n, let f_n be an increasing function that maps $[0,\infty)$ into itself. Show that there is an increasing function g that maps $[0,\infty)$ into itself so that

$$\lim_{x\to\infty} \frac{f_n(x)}{g(x)} = 0$$

for each each positive integer n.

6. Let φ be a continuous, strictly increasing function mapping $[0,1]$ into itself.

(a) Show that the hypothesis '$\varphi(1) = 1$ is the only fixed point of φ in $[0,1]$' is equivalent to the hypothesis '$\varphi(x) > x$ for $0 \le x < 1$'.

(b) Prove: If $\varphi(1) = 1$ is the only fixed point of φ in $[0,1]$, then there is a continuous, strictly increasing function ψ mapping $[0,1]$ into itself such that $\psi \circ \psi = \varphi$.

Hint: Let $a_0 = 0$, let $a_1 = \varphi(a_0)$, and for each positive integer k, let $a_{k+1} = \varphi(a_k)$. Let $b_0 = 0$, let $b_1 = a_1/2$, and for each positive integer k, let $b_{k+2} = \varphi(b_k)$. The sequences a_j and b_k may help break the problem into pieces that can be considered separately.