April 29: For Discussion!

For the following problems, unless otherwise specified, assume all vectors are in $\mathbb{C}^n$ for some positive integer, n, and the inner product, $<\cdot,\cdot>$, is the Euclidean inner product.

110. Let $\mathcal{B} = \{w_1, w_2, \cdots, w_n\}$ be an orthonormal set of vectors in $\mathbb{C}^n$.

(a) Prove that $\mathcal{B}$ is basis for $\mathbb{C}^n$, that is, *an orthonormal basis*, and that for any u in $\mathbb{C}^n$

$$u = <w_1, u> w_1 + <w_2, u> w_2 + \cdots + <w_n, u> w_n$$

(b) Prove: for u and v in $\mathbb{C}^n$, $<u, v> \; = \sum_{j=1}^{n} \overline{<w_j, u>} <w_j, v> \; = \sum_{j=1}^{n} <u, w_j> <w_j, v>$

and therefore that $\|u\|^2 = \sum_{j=1}^{n} | <w_j, u> |^2$

111. The Parallelogram Law from Euclidean Geometry is: The sum of the squares of the lengths of the sides of a parallelogram is equal to the sum of the squares of the lengths of the diagonals. If u and v are vectors that form two sides of a parallelogram, then the diagonals are $u + v$ and $u - v$. Prove the vector form of the Parallelogram Law

$$\|u + v\|^2 + \|u - v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right)$$

112. An $n \times n$ matrix is called *unitary* if $U' = U^{-1}$.

(a) For C an $m \times k$ matrix, prove that the columns of C form an orthonormal set if and only if $C'C = I$.

(b) Prove that an $n \times n$ matrix U is unitary if and only if its columns form an orthonormal basis for $\mathbb{C}^n$.

(c) Prove: if U and V are unitary, then U^{-1} and UV are also unitary.

(d) Show that if U is unitary, then the transformation $x \mapsto Ux$ is a rigid motion in the sense that, for v and w vectors in $\mathbb{C}^n$, $<Uv, Uw> \; = \; <v, w>$ and $\|Uv\| = \|v\|$, so for vectors in $\mathbb{R}^n$, the angle between Uv and Uw is the same as the angle between v and w.

113. The Gram-Schmidt algorithm is specifically created to preserve order:

If v_1, v_2, $\cdots$, v_k is an ordered set of vectors in an inner product space $\mathcal{V}$, then applying the Gram-Schmidt algorithm gives an *orthogonal set* of vectors w_1, w_2, $\cdots$, w_k, so that for $1 \leq j \leq k$, the span of $\{v_1, v_2, \cdots, v_j\}$ is the same as span$\{w_1, w_2, \cdots, w_j\}$.

This is especially important in some engineering or differential equations settings. If $\mathcal{V} = L^2([-1, 1])$, then the functions $1, x, x^2, x^3, \cdots$ span $\mathcal{V}$ in the sense that the closure of the set of polynomials in x is $\mathcal{V}$. The usual inner product on $\mathcal{V}$ is $<f, g> \; = \int_{-1}^{1} \overline{f(t)} g(t)\, dt$, and the *Legendre polynomials* are the orthonormal basis obtained by using Gram-Schmidt on the set of monomials, in the given order, so that the k^{th} Legendre polynomial is a polynomial of degree $k - 1$.

For $\mathcal{V}$ an inner product space, let v_1, $v_2, \cdots$, v_k be an ordered set of vectors in $\mathcal{V}$. For $1 \leq j \leq k - 1$, let P_j be the orthogonal projection of $\mathcal{V}$ onto span$\{v_1, \cdots, v_j\}$. Let $w_1 = v_1$, let $w_2 = v_2 - P_1(v_2)$, and more generally, for $j < k$, let $w_{j+1} = v_{j+1} - P_j(v_{j+1})$. Prove that $\{w_1, w_2, \cdots, w_k\}$ is an orthogonal set of vectors such that, for $1 \leq j \leq k$, the span of $\{v_1, v_2, \cdots, v_j\}$ is the same as span$\{w_1, w_2, \cdots, w_j\}$. In other words, the ordered set $\{w_1, w_2, \cdots, w_k\}$ is the same set as produced by the Gram-Schmidt process.

114. Let M be the hyperplane in $\mathbb{C}^4$ with equation $a+b-c+2d=0$. Find the matrix (with respect to the usual basis) for the orthogonal projection of $\mathbb{C}^4$ onto M. Use it to find the point of M closest to $(1,1,1,1)$.

115. Let U be an $n \times n$ complex matrix that is unitary.

(a) Prove that if λ is an eigenvalue of U, then $|\lambda| = 1$.

(b) Prove that the determinant of U has absolute value 1.

116. Let $\mathcal{V}$ be an inner product space and let $W \neq (0)$ be a subspace of $\mathcal{V}$. Let P be an operator on $\mathcal{V}$ with range$(P) = W$ and $P^2 = P$.

(a) Show that there is v in $\mathcal{V}$ such that $\|Pv\| \geq \|v\|$.

(b) Show that P is the orthogonal projection of $\mathcal{V}$ onto W if and only if $\|Pv\| \leq \|v\|$ for all v in $\mathcal{V}$.

117. Find unitary matrix U and upper triangular matrix T so that $U^{-1}AU = T$ where

$$A = \begin{pmatrix} 1 & -2 & 2 & 1 \\ 0 & -5 & -2 & 3 \\ 0 & 2 & -1 & -1 \\ 0 & -8 & -4 & 5 \end{pmatrix}$$

118. Find all 5×5 matrices N that are both nilpotent and Hermitian.

119. The 5×5 matrix S is Hermitian and v is an eigenvector for S with eigenvalue -3.
The vector w is perpendicular to v. Prove that Sw is also perpendicular to v.

120. Prove that the product of two Hermitian matrices is Hermitian if and only if the matrices commute.

121. (a) Let B be a Hermitian matrix and let $A = B^2$. Prove that if λ is an eigenvalue of A, then λ is real and $\lambda \geq 0$.

(b) A converse of part (a):
Let C be a Hermitian matrix all of whose eigenvalues are non-negative real numbers. Prove that there is a Hermitian matrix B, all of whose eigenvalues are non-negative real numbers, such that $B^2 = C$.

(c) The eigenvalues of $C = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}$ are 1 and 9. Find a Hermitian matrix B, all of whose eigenvalues are non-negative, such that $B^2 = C$.

122. Let T be a normal matrix on the inner product space.
Prove that T is Hermitian if and only if all the eigenvalues of T are real and that T is unitary if and only if all the eigenvalues have modulus 1.

123. Let N be the matrix $N = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$

(a) Show that N is a normal matrix.

(b) Find a unitary matrix U that diagonalizes N.

124. Let $\mathcal{V}$ be the vector space of $n \times n$ complex matrices. Make $\mathcal{V}$ into an inner product space by defining the inner product of two $n \times n$ complex matrices A and B to be $<A, B> = \mathrm{tr}(A^*B)$. For M a fixed $n \times n$ matrix, let T_M be the linear transformation on $\mathcal{V}$ defined by $T_M(A) = MA$. Prove that T_M is unitary on $\mathcal{V}$ if and only if M is a unitary matrix.

125. For T a linear transformation on an inner product space, prove that T is normal if and only if there are Hermitian matrices T_1 and T_2 that commute with each other such that $T = T_1 + iT_2$.

126. Let C and D be $n \times n$ matrices.

(a) Prove that the nullspace of D is a subset of the nullspace of CD.

(b) Prove that the range of CD is a subset of the range of C.

(c) Use the results of (a) and (b) to prove that

$$\mathrm{rank}(CD) \le \mathrm{rank}(C) \qquad \textit{and} \qquad \mathrm{rank}(CD) \le \mathrm{rank}(D).$$

127. Let N be a nilpotent matrix of order k. Prove that $I + N$ is invertible and that

$$(I + N)^{-1} = I - N + N^2 - N^3 + \cdots + (-1)^{k-1}N^{k-1}$$

128. Let T be a linear transformation on a finite dimensional vector space $\mathcal{V}$ that has characteristic polynomial

$$f = (x - c_1)^{d_1}(x - c_2)^{d_2} \cdots (x - c_k)^{d_k}$$

and minimal polynomial

$$p = (x - c_1)^{r_1}(x - c_2)^{r_2} \cdots (x - c_k)^{r_k}$$

Let W_i be the null space of $(T - c_iI)^{r_i}$.

(a) Prove that W_i is an invariant subspace for T.

(b) Letting T_i denote the restriction of T to the invariant subspace W_i, show that $T_i - c_iI$ is nilpotent on W_i and find its order of nilpotence.

(c) Find the minimal polynomial of T_i, the characteristic polynomial of T_i, and the dimension of W_i.

129. Let k and ℓ be positive integers with $k+\ell=n$ and suppose $\mathcal{V}$ is an n-dimensional vector space over the field F. Suppose the sets $\mathcal{B}_1=\{u_1,u_2,\cdots,u_k\}$ and $\mathcal{B}_2=\{v_1,v_2,\cdots,v_\ell\}$ are sets of vectors for which $\mathcal{B}_1\cup\mathcal{B}_2$ forms a basis for $\mathcal{V}$.
Prove that if $\{a_{ij}\}_{i=1,j=1}^{k\ \ \ell}$ are numbers in F and

$$w_j = v_j + \sum_{i=1}^{k} a_{ij}u_i \quad \text{for} \quad 1\le j\le \ell$$

then the set $\mathcal{B}_1\cup\mathcal{B}_3$ also forms a basis for $\mathcal{V}$ where $\mathcal{B}_3=\{w_1,w_2,\cdots,w_\ell\}$.

A Related Topic Not Covered in Math 55400

Definition: Let $\mathcal{V}$ be a real or complex vector space and let K be a non-empty set in $\mathcal{V}$. *The set K is convex* if for each p and q in K and each real number t with $0\le t\le 1$, the point $tp+(1-t)q$ is also in K.

130. Suppose $\mathcal{V}$ is a real or complex vector space and suppose, for some positive integer ℓ, the sets K_1, K_2, $\cdots$, and K_ℓ are convex sets in V.

Prove: If $\bigcap_{j=1}^{\ell} K_j$ is non-empty, then it is a convex set.

131. Suppose V is a real or complex vector space and suppose the set K is a convex subset of V.

Let f be the function defined for x in V by $f(x)=v_0+Tx$ for v_0 a vector in V and T a linear transformation of V into V. (The function f is an example of an *affine map.*)
Prove that $f(K)$ is a convex set in V also.

Definition: Let f be a non-zero linear functional on $\mathbb{R}^n$ and let c be a real number. The set $H=\{x\in\mathbb{R}^n\ : f(x)\le c\}$ is called a *closed half-space of* $\mathbb{R}^n$. If ℓ is a positive integer and H_1, H_2, $\cdots$, and H_ℓ are closed half spaces in $\mathbb{R}^n$, then the set $\bigcap_{j=1}^{\ell} H_j$ is called a *closed polyhedron* in $\mathbb{R}^n$ if it is non-empty.

132. Prove that a closed polyhedron in $\mathbb{R}^n$ is a convex set.

133. Let K be a closed polyhedron in $\mathbb{R}^n$, let g be a linear functional on $\mathbb{R}^n$, and let r be a real number. Prove that $K\cap\{x\in\mathbb{R}^n : g(x)=r\}$ is either empty or a convex set.