April 17

94. Let T be the linear transformation on $\mathbb{C}^3$ whose matrix with respect to the usual basis is

$$\begin{pmatrix} 1 & i & 0 \\ -1 & 2 & -i \\ 0 & 1 & 1 \end{pmatrix}$$

(a) Find the T-annihilator of $(1, 0, 0)$.

(b) Find the T-annihilator of $(1, 0, i)$.

$*$ **95.** Let S be the linear transformation on $\mathbb{R}^3$ represented in the usual basis by the matrix $\begin{pmatrix} 2 & -6 & 3 \\ 3 & -7 & 3 \\ 6 & -12 & 5 \end{pmatrix}$

If p is the minimal polynomial for a matrix, Theorem 12 from class and the text uses the factorization $p = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ where k and $r_1, \cdots, r_k$ are all positive integers.

(a) Express the minimal polynomial for S as $p = p_1^{r_1} p_2^{r_2}$ where p_1 and p_2 are monic, irreducible polynomials over $\mathbb{R}$.

(b) For both $j = 1$ and $j = 2$, find a basis $\mathcal{B}_j$ for W_j, the null space of $p_j(S)$.

(c) Find the matrices for S_1 and S_2, the restrictions of S to W_1 and W_2, with respect to these bases, and also find the matrix for S with respect to the basis $\mathcal{B} = \{\mathcal{B}_1, \mathcal{B}_2\}$.

$**$ **96.** Let T be the linear transformation on $\mathbb{R}^4$ represented in the usual basis by the matrix $\begin{pmatrix} 3 & -2 & 2 & 2 \\ 2 & -1 & 1 & -1 \\ -5 & 2 & -4 & -6 \\ 2 & 0 & 2 & 2 \end{pmatrix}$

Note that $\mathbb{R}^4$ is a real vector space, not a complex vector space.

(a) Find the minimal polynomial of T.

(b) Find the characteristic polynomial of T.

(c) Factor each of these polynomials as a product of monic irreducible polynomials over $\mathbb{R}$.

(d) Using Theorem 12 and your answer to (c), identify k, $r_1, \cdots, r_k$ and polynomials $p_1, \cdots, p_k$ as in the theorem.

(e) Using the notation of Theorem 12, find a basis for each of the subspaces, W_1, $\cdots$, W_k.

(f) For each j, with $1 \le j \le k$, using the notation of Theorem 12, find the matrix for T_1, $\cdots$, T_k, each with respect to the appropriate basis found above.

(g) Find the matrix for T with respect to the basis for $\mathbb{R}^4$ that comes from combining the bases for W_1, $\cdots$, W_k.

$*$ **97.** Let U be the linear transformation on $\mathbb{R}^3$ represented in the usual basis by the matrix $\begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix}$

Show that there are a diagonalizable operator D and a nilpotent operator N on $\mathbb{R}^3$ so that $U = D + N$ and $DN = ND$. Find the matrices for D and N in the usual basis for $\mathbb{R}^3$.

98. Let T be a linear transformation on the finite dimensional vector space $\mathcal{V}$.

(a) Prove that if T^2 has a cyclic vector, then T has a cyclic vector.

(b) Is the converse true? Either give a proof or a counterexample to show that your answer is correct.

$*$ **99.** Let N be a nilpotent linear transformation on the n-dimensional vector space $\mathcal{V}$.

(a) Prove: N has a cyclic vector if and only if $N^{n-1} \neq 0$.

(b) If v is a vector in $\mathcal{V}$ for which $N^{n-1}v \neq 0$, what is the matrix for N with respect to the basis v, Nv, $\cdots$, $N^{n-1}v$.

100. Prove that if A and B are 3×3 matrices over the field F, then A and B are similar if and only if they have the same minimal polynomials and the same characteristic polynomials. Give an example that shows this is not a theorem for 4×4 matrices.

$*$ **101.** Let C be a linear transformation on a finite dimensional vector space $\mathcal{V}$.

(a) Prove: If C does not have a cyclic vector, there is an transformation G that commutes with C, but G is not a polynomial in C.

(b) Prove: If C has a cyclic vector, every transformation that commutes with C is a polynomial in C.

In other words, C has a cyclic vector if and only if every transformation that commutes with C is a polynomial in C.

102. (a) Let A be a linear transformation on the vector space $\mathcal{V}$
and let $v_1, v_2, \cdots, v_k$ be vectors in $\mathcal{V}$.

Prove: If the set $\{Av_1, Av_2, \cdots, Av_k\}$ is a linearly independent set,
then the set $\{v_1, v_2, \cdots, v_k\}$ is also linearly independent.

(b) Show that the converse of the statement in part (a) is false: that is, find a linear transformation T on a vector space $\mathcal{V}$ and a set $\{v_1, v_2, \cdots, v_k\}$ of vectors in $\mathcal{V}$ that are linearly independent, but the set $\{T(v_1), T(v_2), \cdots, T(v_k)\}$ is linearly dependent.