April 10

83. Let $\mathcal{V}$ be an n-dimensional vector space over the field F. Show that if M is any subspace of $\mathcal{V}$, there is a subspace L of $\mathcal{V}$ for which $M \oplus L = \mathcal{V}$. Indeed, if $\mathcal{V}$ is $\mathbb{R}^n$ or $\mathbb{C}^n$, and $0 < \dim(M) < n$, show that there are infinitely many such subspaces.

$*$ **84.** Let $\mathcal{V}$ be an n-dimensional vector space over the field F and let $W_1, W_2, \cdots, W_k$ be subspaces of $\mathcal{V}$ such that

$$\mathcal{V} = W_1 + W_2 + \cdots + W_k \quad \text{and} \quad \dim(V) = \dim(W_1) + \dim(W_2) + \cdots + \dim(W_k)$$

Prove that this means $\mathcal{V} = W_1 \oplus W_2 \oplus \cdots \oplus W_k$.

$*$ **85.** Let E be an $n \times n$ matrix over the field F such that $E^2 = E$.

(a) Show that $I - E$ is also a projection matrix.

(b) If E is described as the projection onto R along N, what is the description of $I - E$?

(c) Let $Q = \begin{pmatrix} -1 & 2 & -2 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix}$

Show that Q is a projection and describe Q as in part (b).

$*$ **86.** Consider the statement: "If a diagonalizable operator has only eigenvalues 0 and 1, then it is a projection." If it is true, prove it; if it is false, find an example.

87. Let $E_1, E_2, \cdots, E_k$ be projection matrices on $\mathbb{R}^n$ for which $E_1 + E_2 + \cdots + E_k = I$. Use the trace function to show that $E_iE_j = 0$ for $i \neq j$.

88. Let E be a projection on the real vector space $\mathcal{V}$. Prove that $I + E$ is invertible and find $(I + E)^{-1}$.

89. Suppose $\mathcal{V}$ is a vector space over the field F and for $j = 1, \cdots, k$ the subspaces W_j satisfy

$$\mathcal{V} = W_1 \oplus W_2 \oplus \cdots \oplus W_k$$

Let T be a linear transformation on $\mathcal{V}$ for which the subspaces W_j are invariant for $j = 1, \cdots, k$, let T_j be the restriction of T to W_j, let A_j be the matrix for T_j with respect to the basis $\mathcal{B}_j$ for W_j, and let A be the matrix for T with respect to the basis $\mathcal{B} = \{\mathcal{B}_1, \cdots, \mathcal{B}_k\}$ for $\mathcal{V}$.

(a) Show that $\det(A) = \det(A_1)\det(A_2)\cdots\det(A_k)$.

(b) Prove that if f_j is the characteristic polynomial of T_j and A_j, then the characteristic polynomial of T and A is f, the product of the f_j's.

(c) Prove that the minimal polynomial of T and A is the least common multiple of the minimal polynomials of the T_j's.

90. Let P and Q be projections on the real vector space $\mathcal{V}$ for which $PQ = QP$.
Prove that PQ is also a projection and find the range and nullspace of PQ.

91. Suppose $\mathcal{V}$ is a vector space over the field F and E and T are, respectively, a projection and a linear transformation on $\mathcal{V}$.

(a) Show that the range of E is invariant for T if and only if $ETE = TE$.

(b) Show that the range and nullspace of E are *both* invariant for T if and only if $TE = ET$.

(c) Which operators commute with *every* projection on $\mathcal{V}$?

$*$ **92.** Let $G = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$

(It might be helpful to observe that 1 is an eigenvalue of G.)

(a) Find the characteristic and minimal polynomials for G and explain how you know that G is diagonalizable over the field $\mathbb{R}$.

(b) Find eigenspaces W_1, W_2, and W_3 that are invariant subspaces for G giving a direct sum decomposition of $\mathbb{R}^4$ as $W_1 \oplus W_2 \oplus W_3$.

(c) Find projections E_1, E_2, and E_3 so that $E_1 + E_2 + E_3 = I$, $E_iE_j = 0$ for $i \neq j$ and $G = aE_1 + bE_2 + cE_3$ for some real numbers a, b, and c.

$**$ **93.** Let $\mathcal{V}$ be an n-dimensional vector space, suppose that $c_1, c_2, \cdots, c_k$ are distinct scalars in the field F, and suppose $E_1, E_2, \cdots, E_k$ are projections on $\mathcal{V}$ such that $E_iE_j = 0$ for $i \neq j$ and $I = \sum_{j=1}^k E_j$.

Let $T = c_1E_1 + c_2E_2 + \cdots + c_kE_k$.

(a) Find (and prove) a simple expression for T^2 in terms of the E_j's.

(b) For p a polynomial, find (and prove) a simple expression for $p(T)$ in terms of the E_j's.

(c) Find the minimal polynomial for T and find characteristic polynomial for T.