April 3

74. Let $A = \begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix}$

Show that A is not similar, over the field $\mathbb{R}$, to a diagonal matrix.
Is A similar to a diagonal matrix over $\mathbb{C}$?

∗ **75.** Let B be an $n \times n$ matrix over the field F and let v be a vector in F^n.

(a) Prove that the set

$$J_v = \{p \in F[x] : p(B)v = 0\}$$

is an ideal in $F[x]$.

(b) Prove that the monic generator, q, of J_v must divide the minimal polynomial of B and, therefore, it must divide the characteristic polynomial of B.

(c) Conclude: If the degree of q is n, then q is actually the characteristic polynomial of B.

∗ **76.** Let $C = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

Choose a non-zero vector v in $\mathbb{C}^4$ and find a monic polynomial q for which $q(C)v = 0$.
Use your answer, q, to find the characteristic polynomial for C.

∗ **77.** Let C and D be $n \times n$ matrices over the field F.

(a) Prove that if $I - CD$ is invertible, then $I - DC$ is also invertible and

$$(I - DC)^{-1} = I + D(I - CD)^{-1}C$$

(b) Use this result to show that CD and DC have the same eigenvalues over the field F.

78. Let N be a linear transformation on an n-dimensional vector space $\mathcal{V}$ over the field F.
Prove: if $N^k = 0$ for some positive integer k, then $N^n = 0$.

∗ **79.** Let $E = \begin{pmatrix} 0 & 1 & 0 \\ 2 & -2 & 2 \\ 2 & -3 & 2 \end{pmatrix}$

Either find an upper triangular matrix, F, that is similar to E over the field, $\mathbb{R}$, of real numbers, or prove that E is not similar to any upper triangular matrix over $\mathbb{R}$.

$*$ **80.** Let $G = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$

(a) Suppose M is an invariant subspace for an operator H on a vector space $\mathcal{V}$. Show that the eigenvalues of the restriction of H to M are also eigenvalues of H on $\mathcal{V}$.

(b) Find all the 1-dimensional invariant subspaces for G.

(c) Find all the 2-dimensional invariant subspaces for G.

81. Find an invertible matrix S so that $S^{-1}PS$ and $S^{-1}QS$ are both diagonal where P and Q are the real matrices

(a) $P = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$ and $Q = \begin{pmatrix} 3 & -8 \\ 0 & -1 \end{pmatrix}$

(b) $P = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}$

$**$ **82.** Look back at exercise 76. You chose a vector 'at random' to use for finding the polynomial q that worked for your vector and the given matrix C.
Probably the degree of the polynomial you found from using your vector was 4.

Suppose A is a 4×4 matrix with complex entries.

(a) For which v in $\mathbb{C}^4$ will A^4v, A^3v, A^2v, Av and Iv be linearly dependent? Why?

(b) For which v in $\mathbb{C}^4$ will Av and Iv be linearly dependent? Why?

(c) For which v in $\mathbb{C}^4$ will A^2v, Av and Iv be linearly dependent? Why?

(d) For which v in $\mathbb{C}^4$ will A^3v, A^2v, Av and Iv be linearly dependent? Why?

(e) Explain why it was extremely likely that choosing a vector 'at random' from R^4 would give a polynomial of degree 4.