March 27

65. Let A be an $n \times n$ matrix with entries in K, a commutative ring with identity. Prove that $\det(A) = \det(A^t)$, where A^t is the transpose of A.

66. Do *NOT* use determinants in doing this exercise! An $m \times n$ matrix $A = (a_{ij})$ is said to be *lower triangular* if $a_{ij} = 0$ for $i < j$ and *upper triangular* if $a_{ij} = 0$ for $i > j$.

(a) Prove: If A is a lower triangular $k \times m$ matrix and B is a lower triangular $m \times n$ matrix, then AB is a lower triangular $k \times n$ matrix.

(b) Prove that a lower triangular $n \times n$ matrix A is invertible if and only if the diagonal entries of A are all non-zero.

(c) Show that if A is a lower triangular $n \times n$ matrix that is invertible, then A^{-1} is also a lower triangular matrix.

67. Recall that we (inductively, about March 4 or 6) defined several determinant functions for $n \times n$ matrices by the formula

$$(*) \qquad \det(A) = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} \det(A(i|j))$$

where $A(i|j)$ is the $(n-1) \times (n-1)$ matrix obtained from A by deleting the i^{th} row and the j^{th} column. But, since we have proved the determinant function is unique, we conclude they are all the same as the function 'det'.

The formula $(*)$ is called *finding the determinant by expansion along the j^{th} column* of A. The scalar $C_{ij} = (-1)^{i+j} \det A(i|j)$ is called the i, j *cofactor of* A. From the above, we can easily see that $\det(A) = \sum_{i=1}^{n} A_{ij} C_{ij}$. The *adjugate matrix of* A (also sometimes called the 'classical adjoint' of A) is the matrix $\mathrm{adj}(A) = B$ where $B_{ij} = C_{ji}$, the transpose of the matrix of cofactors.

(a) Let A be an $n \times n$ matrix over the ring K and $(b_1 b_2 \cdots b_n)$ be a row vector in K^n. Identify an $n \times n$ matrix over K whose determinant is $\sum_{i=1}^{n} b_i C_{ij}$.

(b) Using part (a) above, prove that for any ring K, the adjugate matrix satisfies $\mathrm{adj}(A)A = (\det(A))I$ by recognizing the expansion for each entry as a determinant of a specific matrix.

(c) Show that if K is actually a field $\mathbb{F}$, then the matrix A is invertible if and only if the determinant of A is not zero and for A with non-zero determinant

$$A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A)$$

68. Use the ideas of the exercise above, in particular by recognizing a sum as the determinant of a particular matrix, prove

Cramer's Rule: If A is an $n \times n$ matrix over a field $\mathbb{F}$ and $\det(A)$ is not zero, then the unique solution of $AX = b$ is $x_j = \det(B_j)/\det(A)$ where B_j is the matrix obtained by replacing the jth column of A by b, that is

$$B_j = \begin{pmatrix} C_1 & \cdots & C_{j-1} & b & C_{j+1} & \cdots & C_n \end{pmatrix}$$

where the columns of A are C_1, C_2, $\cdots$, C_n.

∗ **69.** An $n \times n$ real matrix $A = (a_{ij})$ is said to be *symmetric* if $a_{ij} = a_{ji}$ for $i, j = 1, \cdots, n$, that is, if $A^t = A$. For this problem, suppose A and B are symmetric $n \times n$ real matrices.

(a) Prove: If A and B commute, that is, $AB = BA$, then AB is also a symmetric matrix.

(b) Give an example of two symmetric real matrices whose product is not symmetric.

(c) Prove: If A is a real $n \times n$ symmetric matrix that is invertible, then A^{-1} is also symmetric.

∗ **70.** Let f and g be monic polynomials over the field $\mathbb{C}$. Assume the Fundamental Theorem of Algebra to do this exercise.

(a) Prove that the g.c.d. of f and g is 1 if and only if f and g have no common roots.

(b) Let f be of degree k and $f(x) = (x - c_1)(x - c_2) \cdots (x - c_k)$. Prove: the c_j are distinct complex numbers if and only if f and Df have no common roots. (Here D is the formal derivative transformation on polynomials, which you may assume satisfies the product rule.)

(c) Find monic real polynomials p and q, each of degree three, that have no common (real) roots but the g.c.d. of p and q over $\mathbb{R}$ is not 1.

∗ **71.** Let B be an $n \times n$ matrix over the ring K that has block diagonal form:

$$B = \begin{pmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & B_3 \end{pmatrix}$$

where B_j is a $d_j \times d_j$ matrix and $n = d_1 + d_2 + d_3$.

Prove that $\det(B) = \det(B_1)\det(B_2)\det(B_3)$.

(Although the problem asserts this for 3 blocks, it is true for any finite number of blocks.)

∗ **72.** Let α_1, α_2, $\cdots$, and α_n be elements of the ring K and let C be the $n \times n$ matrix over K that has entries $c_{j,j+1} = 1$ for $1 \leq j \leq n-1$, $c_{n,j} = \alpha_j$ for $1 \leq j \leq n$ and $c_{ij} = 0$ otherwise. That is,

$$C = \begin{pmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \end{pmatrix}$$

Find the characteristic polynomial of C.

$**$ **73.** In class, we noted that if P, Q, and R are, respectively, $r \times r$, $r \times s$, and $s \times s$ real or complex matrices and, for $n = r + s$, we let G be the $n \times n$ matrix with block form $G = \begin{pmatrix} P & Q \\ 0 & R \end{pmatrix}$, then $\det(G) = \det(P)\det(R)$.

(a) Prove that if G is an $n \times n$ real or complex matrix with block form $G = \begin{pmatrix} P & 0 \\ Q & R \end{pmatrix}$, then $\det(G) = \det(P)\det(R)$.

(b) Suppose A, B, C, D are commuting $n \times n$ real or complex matrices.

Factor $H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ into a product of upper and lower triangular block matrices

or otherwise show, that $\det(H) = \det(AD - BC)$

(c) Give an example of $n \times n$ matrices A, B, C, and D over $\mathbb{R}$ for which

$H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, and $\det(H) \neq \det(AD - BC)$.