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37. Let V be an n-dimensional vector space and let $\mathcal{B}_1 = \{v_1, v_2, \cdots, v_n\}$ and $\mathcal{B}_2 = \{w_1, w_2, \cdots, w_n\}$ be bases for V. Let T be a linear transformation of V into itself. We say the *$n \times n$ matrix M_1 with entries (a_{ij}) for $1 \leq i, j \leq n$ is the matrix for the transformation T with respect to the basis $\mathcal{B}_1$* if, for each $1 \leq j \leq n$ we have $T(v_j) = a_{1j}v_1 + a_{2j}v_2 + \cdots + a_{nj}v_n$. In the same way, there is an $n \times n$ matrix M_2 with entries (b_{ij}) for $1 \leq i, j \leq n$ that is the matrix for T with respect to the basis $\mathcal{B}_2$ so that for each $1 \leq j \leq n$ we have $T(w_j) = b_{1j}w_1 + b_{2j}w_2 + \cdots + b_{nj}w_n$. Prove that, in this situation, there is an invertible matrix S such that $M_2 = S^{-1}M_1S$ and that this *change of basis matrix,* S does not depend on T, but only on the bases $\mathcal{B}_1$ and $\mathcal{B}_2$.

$*$ **38.** Let $W = \text{span}\left\{ v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 3 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 4 \\ 6 \\ 4 \\ 1 \end{pmatrix} \right\}$ Find a basis $\{f_j\}$ for W°.

$**$ **39.** Let M and N be subspaces of the finite dimensional vector space V.

(a) Prove that $(M + N)^\circ = M^\circ \cap N^\circ$.

(b) Prove that $(M \cap N)^\circ = M^\circ + N^\circ$.

$*$ **40.** Prove that linear functionals on subspaces can be extended to the whole space: That is, suppose V is a finite dimensional vector space and W is a subspace of V and suppose g is a linear functional defined on W. Prove that there is a linear functional f defined on all of V for which $f(w) = g(w)$ for all w in the subspace W.

$*$ **41.** Let F be a field of characteristic zero (that is, in a field where no finite sum of 1's is 0) and suppose V is a finite dimensional vector space over F. Show that if v_1, v_2, $\cdots$, v_m is a finite set of non-zero vectors in V, there is a linear functional f on V for which $f(v_j) \neq 0$ for $j = 1, 2, \cdots, m$.

$*$ **42.** Let M and N be subspaces of the finite dimensional vector space V.

(a) Let T be an isomorphism of V onto the vector space W. Prove that $M \subset N$ if and only if $T(M) \subset T(N)$.

(b) Prove that $M \subset N$ implies $N^\circ \subset M^\circ$.

(c) Prove that $N^\circ \subset M^\circ$ implies $M \subset N$.

43. Show that, for $n \geq 2$, the trace functional on $n \times n$ matrices is unique in the following sense: If W is the vector space of $n \times n$ matrices over the field F and f is a linear functional on W such that $f(AB) = f(BA)$ for each A and B in W, then f is a scalar multiple of the trace function.