January 23

$*$ **10.** Let $A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 2 \\ 1 & -1 & 0 & -1 \end{pmatrix}$ let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ and let $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$

where x_1, x_2, x_3, and x_4 and y_1, y_2, y_3, and y_4 are variables whose values are real numbers. Find conditions on Y that ensure the equation $AX = Y$ has solutions. (See problem 4.)

$*$ **11.** Let $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix with real entries a, b, c, and d. Show that there are 2×2 real matrices A and B so that $C = AB - BA$ if and only if $a + d = 0$.

12. Use partitioned (block) matrices to show that if A is an $m \times n$ matrix and B is an $n \times p$ matrix whose k^{th} column is zero, then the k^{th} column of AB is zero.

$*$ **13.** Let $\mathcal{F}$ be a field. Let C be the $m \times p$ matrix $C = AB$ where A and B are, respectively, $m \times n$ and $n \times p$ matrices with entries in the field $\mathcal{F}$.

Prove that the columns of C are linear combinations of the columns of A, that is, specifically, if C_1, C_2, $\cdots$, and C_p are the columns of C, and A_1, A_2, $\cdots$, and A_n are the columns of A, then there are coefficients $\{\beta_{ij}\}$, each in the field $\mathcal{F}$, so that for each i,

$$C_i = \sum_{j=1}^{n} \beta_{ij} A_j$$

$*$ **14.** An $n \times n$ matrix A is said to be an *upper triangular* if every entry below the main diagonal is 0. Prove that such a matrix A is invertible if and only if every entry on the main diagonal is non-zero.

15. Let A be an $n \times n$ matrix. Prove:

(a) If A is invertible and B is an $n \times n$ matrix for which $AB = 0$, then $B = 0$.

(b) If A is not invertible, there is B a non-zero $n \times n$ matrix for which $AB = 0$.

$**$ **16.** Suppose A is a square matrix with entries in a field F partitioned as

$$A = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}$$

where X and Z are square matrices and 0 is a zero matrix.

(a) Find necessary and sufficient conditions on X, Y, and Z so that A is invertible and then find formulas for P, Q, R, and S so that A^{-1} is the block matrix

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

(b) Use your formula to find A^{-1}

when $X = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -3 & 3 \\ 3 & -2 & 5 \end{pmatrix}$, $Y = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ -3 & 1 \end{pmatrix}$, and $Z = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$

17. Suppose A is an $n \times n$ real matrix, u and v are real column and row vectors, respectively, and a is a real number. The matrix

$$X = \begin{pmatrix} A & u \\ v & a \end{pmatrix}$$

is an $(n+1) \times (n+1)$ matrix called a *bordered matrix.*

(a) Show that if A is invertible and $a - vA^{-1}u \neq 0$ then X is invertible by writing

$$X^{-1} = \begin{pmatrix} B & p \\ q & b \end{pmatrix}$$

and finding B, p, q, and b in terms of A, u, v, and a.

(b) Check the formula you found above by using it to find the inverse of

$$X = \begin{pmatrix} 1 & -2 & 2 \\ -1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$