February 23 to 28 The following problems will not be collected but might be helpful in preparing for the midterm

53. Let F be a field and let f be in F^∞, that is, f is a formal power series with coefficients in F. In analogy with evaluating polynomials at scalars from F, for f in F^∞ and a in F, define $f(a)$ in F^∞ by:

$$\text{For} \quad f = (f_0, f_1, f_2, f_3, \cdots) \qquad \text{let} \quad f(a) = (f_0, f_1 a, f_2 a^2, f_3 a^3, f_4 a^4, \cdots)$$

In F^∞ for F a subfield of $\mathbb{C}$, let *exp* and, for a in F, $exp(a)$ be the formal power series

$$exp = (1, 1, (2!)^{-1}, (3!)^{-1}, \cdots) \quad \text{and} \quad exp(a) = (1, a, a^2/2!, a^3/3!, a^4/4!, \cdots)$$

Using the definition of products in F^∞ and the binomial theorem, prove that, for a and b in F,

$$exp(a)exp(b) = exp(a+b)$$

54. Let F be a field and let $F[x]$ be the algebra of polynomials over F.

(a) Prove: If $a \neq 0$ and b are elements of F, the polynomials 1, $ax+b$, $(ax+b)^2$, $(ax+b)^3$, $\cdots$, form a basis for $F[x]$.

(b) More generally, show that if h is a polynomial in F of degree at least 1 then the mapping $T(f) = f(h)$ is a linear transformation of $F[x]$ into itself.

(c) Show that the transformation T in part (b) is an isomorphism of $F[x]$ onto $F[x]$ if and only if h has degree 1.

55. Let F be a field and let $F[x]$ be the algebra of polynomials over F.

(a) Prove that the intersection of any number of ideals in $F[x]$ is also an ideal in $F[x]$.

(b) Let f_1, f_2, $\cdots$, f_k be polynomials in $F[x]$ and let J be the ideal generated by $\{f_1, f_2, \cdots, f_k\}$. Show that J is the intersection of all of the ideals in $F[x]$ that contain all of the f_j for $j = 1, \cdots, k$

56. Let f and g be monic polynomials over the field $\mathbb{C}$. Assume the Fundamental Theorem of Algebra to do this exercise.

(a) Prove that the g.c.d. of f and g is 1 if and only if f and g have no common roots.

(b) Let f be of degree k and $f(x) = (x - c_1)(x - c_2)\cdots(x - c_k)$. Prove: the c_j are distinct complex numbers if and only if f and Df have no common roots. (Here D is the formal derivative transformation on polynomials, which you may assume satisfies the product rule.)

(c) Find monic real polynomials p and q, each of degree three, that have no common (real) roots but the g.c.d. of p and q over $\mathbb{R}$ is not 1.

57. Do *NOT* use determinants to do this exercise! An $m \times n$ matrix $A = (a_{ij})$ is said to be *lower triangular* if $a_{ij} = 0$ for $i < j$ and *upper triangular* if $a_{ij} = 0$ for $i > j$.

(a) Prove: If A is a lower triangular $k \times m$ matrix and B is a lower triangular $m \times n$ matrix, then AB is a lower triangular $k \times n$ matrix.

(b) Prove that a lower triangular $n \times n$ matrix A is invertible if and only if the diagonal entries of A are all non-zero.

(c) Show that if A is a lower triangular $n \times n$ matrix that is invertible, then A^{-1} is also a lower triangular matrix.

58. An $n \times n$ matrix $T = (t_{ij})$ is said to be a *Toeplitz matrix* if $t_{ij} = t_{i+1,j+1}$ for $1 \leq i, j < n$.

(a) Prove: If S and T are a lower triangular $n \times n$ Toeplitz matrices, then ST is a lower triangular Toeplitz matrix also.

(b) Give an example to show that if S and T are both $n \times n$ Toeplitz matrices, then it is not necessarily the case that ST is a Toeplitz matrix.

(c) Prove: If $T = (t_{ij})$ is a lower triangular $n \times n$ Toeplitz matrix with $t_{11} \neq 0$, then T is invertible and T^{-1} is also a Toeplitz matrix.

(d) Let T be the 4×4 Toeplitz matrix with $t_{1,1} = 1$, $t_{2,1} = -2$, and $t_{3,1} = 1$ with $t_{4,1} = t_{1,2} = t_{1,3} = t_{1,4} = 0$. Find T^{-1}.

(e) Let T be the $n \times n$ Toeplitz matrix with $t_{1,1} = 1$, $t_{2,1} = -2$, and $t_{3,1} = 1$ and $t_{i,j} = 0$ for $i - j \neq 0, 1,$ or 2. Make a conjecture for T^{-1}. Can you prove your conjecture?

59. An $n \times n$ real matrix $A = (a_{ij})$ is said to be *symmetric* if $a_{ij} = a_{ji}$ for $i, j = 1, \cdots, n$, that is, if $A^t = A$. For this problem, suppose A and B are symmetric $n \times n$ real matrices.

(a) Prove: If A and B commute, that is, $AB = BA$, then AB is also a symmetric matrix.

(b) Give an example of two symmetric real matrices whose product is not symmetric.

(c) Prove: If A is a real $n \times n$ symmetric matrix that is invertible, then A^{-1} is also symmetric.