February 15

$*$ **33.** Show that if A and B are $n \times n$ matrices over the field F, then $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. Using this, show that similar matrices have the same trace. (Note: A and B are *similar* if there is an invertible matrix S so that $B = S^{-1}AS$.)

34. Show that $AB - BA = I$ is impossible for $n \times n$ matrices of complex numbers. Can $AB - BA = I$ when A and B are 3×3 matrices over the field F_3?

$*$ **35.** Let V be the vector space of polynomial functions of $\mathbb{R}$ into $\mathbb{R}$ of degree 2 or less:

$$p(x) = c_0 + c_1 x + c_2 x^2 \qquad \text{for } c_0, c_1, \text{ and } c_2 \in \mathbb{R}$$

Define linear functionals on V by

$$f_1(p) = \int_0^1 p(x)\,dx \qquad f_2(p) = \int_0^2 p(x)\,dx \qquad f_3(p) = \int_{-1}^0 p(x)\,dx$$

Show that $\{f_1, f_2, f_3\}$ is a basis for V^* by exhibiting the basis of V for which these linear functionals form the dual basis for.

$*$ **36.**

$$\text{Let} \quad W = \text{span}\left\{ v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \; v_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 3 \\ 1 \end{pmatrix}, \; v_3 = \begin{pmatrix} 1 \\ 4 \\ 6 \\ 4 \\ 1 \end{pmatrix} \right\} \qquad \text{Find a basis } \{f_j\} \text{ for } W^\circ.$$

$**$ **37.** Let M and N be subspaces of the finite dimensional vector space V.

(a) Prove that $(M + N)^\circ = M^\circ \cap N^\circ$.

(b) Prove that $(M \cap N)^\circ = M^\circ + N^\circ$.

$*$ **38.** Prove that linear functionals on subspaces can be extended to the whole space: That is, suppose V is a finite dimensional vector space and W is a subspace of V and suppose g is a linear functional defined on W. Prove that there is a linear functional f defined on all of V for which $f(w) = g(w)$ for all w in the subspace W.

$*$ **39.** Let F be a field of characteristic zero (that is, in a field where no finite sum of 1's is 0) and suppose V is a finite dimensional vector space over F. Show that if $v_1, v_2, \cdots, v_m$ is a finite set of non-zero vectors in V, there is a linear functional functional f on V for which $f(v_j) \neq 0$ for $j = 1, 2, \cdots, m$.

$**$ **40.** Let M and N be subspaces of the finite dimensional vector space V.

(a) Let T be an isomorphism of V onto the vector space W. Prove that $M \subset N$ if and only if $T(M) \subset T(N)$.

(b) Prove that $M \subset N$ implies $N^\circ \subset M^\circ$.

(c) Prove that $N^\circ \subset M^\circ$ implies $M \subset N$.

41. Show that the trace functional on $n \times n$ matrices is unique in the following sense: If W is the vector space of $n \times n$ matrices over the field F and f is a linear functional on W such that $f(AB) = f(BA)$ for each A and B in W, then f is a scalar multiple of the trace function.

Definition: Let V be a real or complex vector space and let K be a non-empty set in V. *The set K is convex* if for each p and q in K and each real number t with $0 \leq t \leq 1$, the point $tp + (1-t)q$ is also in K.

42. Suppose V is a real or complex vector space and suppose, for some positive integer ℓ, the sets K_1, K_2, $\cdots$, and K_ℓ are convex sets in V.

Prove: If $\bigcap_{j=1}^{\ell} K_j$ is non-empty, then it is a convex set.

43. Suppose V is a real or complex vector space and suppose the set K is a convex subset of V. Let f be the function defined for x in V by $f(x) = v_0 + Tx$ for v_0 a vector in V and T a linear transformation of V into V. (The function f is an example of an *affine map.*)

Prove that $f(K)$ is a convex set in V also.

Definition: Let f be a non-zero linear functional on $\mathbb{R}^n$ and let c be a real number. The set $H = \{x \in \mathbb{R}^n \ : f(x) \leq c\}$ is called a *closed half-space of* $\mathbb{R}^n$*.* If ℓ is a positive integer and H_1, H_2, $\cdots$, and H_ℓ are closed half spaces in $\mathbb{R}^n$, then the set $\bigcap_{j=1}^{\ell} H_j$ is called a *closed polyhedron* in $\mathbb{R}^n$ if it is non-empty.

44. Prove that a closed polyhedron in $\mathbb{R}^n$ is a convex set.

45. Let K be a closed polyhedron in $\mathbb{R}^n$, let g be a linear functional on $\mathbb{R}^n$, and let r be a real number. Prove that $K \cap \{x \in \mathbb{R}^n : g(x) = r\}$ is either empty or a convex set.