January 18

$*$ **10.**

$$\text{Let} \quad A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 2 \\ 1 & -1 & 0 & -1 \end{pmatrix} \quad \text{let} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \text{and let} \quad Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

where x_1, x_2, x_3, and x_4 and y_1, y_2, y_3, and y_4 are variables whose values are real numbers. Find conditions on Y that ensure the equation $AX = Y$ has solutions. (See problem 4.)

$*$ **11.**
Let $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix with real entries a, b, c, and d. Show that there are 2×2 real matrices A and B so that $C = AB - BA$ if and only if $a + d = 0$.

$*$ **12.**
Let $\mathcal{F}$ be a field. Let C be the $m \times p$ matrix $C = AB$ where A and B are, respectively, $m \times n$ and $n \times p$ matrices with entries in the field $\mathcal{F}$.

Prove that the columns of C are linear combinations of the columns of A, that is, specifically, if C_1, C_2, $\cdots$, and C_p are the columns of C, and A_1, A_2, $\cdots$, and A_n are the columns of A, then there are coefficients $\{\beta_{ij}\}$, each in the field $\mathcal{F}$, so that for each i,

$$C_i = \sum_{j=1}^{n} \beta_{ij} A_j$$