May 1: For Discussion!

For the following problems, unless otherwise specified, assume all vectors are in $\mathbb{C}^n$ for some positive integer, n, and the inner product, $<\cdot,\cdot>$, is the Euclidean inner product.

114. An $n \times n$ matrix is called *unitary* if $U' = U^{-1}$.

(a) For C an $m \times k$ matrix, prove that the columns of C form an orthonormal set if and only if $C'C = I$.

(b) Prove that an $n \times n$ matrix U is unitary if and only if its columns form an orthonormal basis for $\mathbb{C}^n$.

(c) Prove: if U and V are unitary, then U^{-1} and UV are also unitary.

(d) Show that if U is unitary, then the transformation $x \mapsto Ux$ is a rigid motion in the sense that, for v and w vectors in $\mathbb{C}^n$, $<Uv, Uw> = <v, w>$ and $\|Uv\| = \|v\|$, so for vectors in $\mathbb{R}^n$, the angle between Uv and Uw is the same as the angle between v and w.

115. The Gram-Schmidt algorithm is specifically created to preserve order: If v_1, v_2, $\cdots$, v_k is an ordered set of vectors in an inner product space $\mathcal{V}$, then applying the Gram-Schmidt algorithm gives an *orthogonal set* of vectors w_1, w_2, $\cdots$, w_k, so that for $1 \le j \le k$, the span of $\{v_1, v_2, \cdots, v_j\}$ is the same as span$\{w_1, w_2, \cdots, w_j\}$. This is especially important in some engineering or differential equations settings.

If $\mathcal{V} = L^2([-1,1])$, then the functions $1, x, x^2, x^3, \cdots$ span $\mathcal{V}$ in the sense that the closure of the set of polynomials in x is $\mathcal{V}$. The usual inner product on $\mathcal{V}$ is $<f, g> = \int_{-1}^{1} \overline{f(t)} g(t)\, dt$, and the *Legendre polynomials* are the orthonormal basis obtained by using Gram-Schmidt on the set of monomials, in the given order, so that the k^{th} Legendre polynomial is a polynomial of degree $k-1$.

For $\mathcal{V}$ an inner product space, let v_1, $v_2, \cdots$, v_k be an ordered set of vectors in $\mathcal{V}$. For $1 \le j \le k-1$, let P_j be the orthogonal projection of $\mathcal{V}$ onto span$\{v_1, \cdots, v_j\}$. Let $w_1 = v_1$, let $w_2 = v_2 - P_1(v_2)$, and more generally, for $j < k$, let $w_{j+1} = v_{j+1} - P_j(v_{j+1})$. Prove that $\{w_1, w_2, \cdots, w_k\}$ is an orthogonal set of vectors such that, for $1 \le j \le k$, the span of $\{v_1, v_2, \cdots, v_j\}$ is the same as span$\{w_1, w_2, \cdots, w_j\}$. In other words, the ordered set $\{w_1, w_2, \cdots, w_k\}$ is the same set as produced by the Gram-Schmidt process.

116. Let M be the hyperplane in $\mathbb{C}^4$ with equation $a + b - c + 2d = 0$. Find the matrix (with respect to the usual basis) for the orthogonal projection of $\mathbb{C}^4$ onto M. Use it to find the point of M closest to $(1, 1, 1, 1)$.

117. Let U be an $n \times n$ complex matrix that is unitary.

(a) Prove that if λ is an eigenvalue of U, then $|\lambda| = 1$.

(b) Prove that the determinant of U has absolute value 1.

118. Let $\mathcal{V}$ be an inner product space and let $W \ne (0)$ be a subspace of $\mathcal{V}$. Let P be an operator on $\mathcal{V}$ with range$(P) = W$ and $P^2 = P$.

(a) Show that there is v in $\mathcal{V}$ such that $\|Pv\| \ge \|v\|$.

(b) Show that P is the orthogonal projection of $\mathcal{V}$ onto W if and only if $\|Pv\| \le \|v\|$ for all v in $\mathcal{V}$.

119. Find unitary matrix U and upper triangular matrix T so that $U^{-1}AU = T$ where

$$A = \begin{pmatrix} 1 & -2 & 2 & 1 \\ 0 & -5 & -2 & 3 \\ 0 & 2 & -1 & -1 \\ 0 & -8 & -4 & 5 \end{pmatrix}$$

120. Find all 5×5 matrices N that are both nilpotent and Hermitian.

121. The 5×5 matrix S is Hermitian and v is an eigenvector for S with eigenvalue -3. The vector w is perpendicular to v. Prove that Sw is also perpendicular to v.

122. Prove that the product of two Hermitian matrices is Hermitian if and only if the matrices commute.

123.

(a) Let B be a Hermitian matrix and let $A = B^2$. Prove that if λ is an eigenvalue of A, then λ is real and $\lambda \geq 0$.

(b) (A converse of part (a).) Let C be a Hermitian matrix all of whose eigenvalues are non-negative real numbers. Prove that there is a Hermitian matrix B, all of whose eigenvalues are non-negative real numbers, such that $B^2 = C$.

(c) The eigenvalues of $C = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}$ are 1 and 9. Find a Hermitian matrix B, all of whose eigenvalues are non-negative, such that $B^2 = C$.

124.

Let N be the matrix $N = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$

(a) Show that N is a normal matrix.

(b) Find a unitary matrix U that diagonalizes N.

125. Let $\mathcal{V}$ be the vector space of $n \times n$ complex matrices. Make $\mathcal{V}$ into an inner product space by defining the inner product of two $n \times n$ complex matrices A and B to be $<A, B> = \text{tr}(A'B)$. For M a fixed $n \times n$ matrix, let T_M be the linear transformation on $\mathcal{V}$ defined by $T_M(A) = MA$. Prove that T_M is unitary on $\mathcal{V}$ if and only if M is a unitary matrix.

126. Let T be a normal matrix on the inner product space. Prove that T is Hermitian if and only if all the eigenvalues of T are real and that T is unitary if and only if all the eigenvalues have modulus 1.

127. For T a linear transformation on an inner product space, prove that T is normal if and only if there are Hermitian matrices T_1 and T_2 that commute with each other such that $T = T_1 + iT_2$.