Notation and Definitions for Material on $\mathbb{R}^n$ and $\mathbb{C}^n$ as Inner Product Spaces

Definition: If A is an $m \times n$ real or complex matrix the *adjoint of* A, denoted by A', is the conjugate transpose of A. That is, if $A = (a_{ij})_{i=1,j=1}^{m\ \ n}$, then $A' = B$ where $B = (b_{k\ell})_{k=1,\ell=1}^{n\ \ m}$ and the entries of B satisfy $b_{ij} = \overline{a_{ji}}$.

Unless otherwise specified, if v is in $\mathbb{R}^n$ or $\mathbb{C}^n$, we will not distinguish, by any notation, between thinking of v as a column vector or an $n \times 1$ matrix. We will also regard a 1×1 matrix as representing the number that is its entry.

Definition: The *usual* or *Euclidean* or *standard* inner product for vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$, denoted $<\cdot,\cdot>$, is defined to be the number

$$<u,v> = u'v \quad \text{for } u \text{ and } v \text{ in } \mathbb{R}^n \text{ or } \mathbb{C}^n$$

In particular, this means for $u = (u_1, u_2, \cdots, u_n)$ and $v = (v_1, v_2, \cdots, v_n)$, then

$$<u,v> = \overline{u_1}v_1 + \overline{u_2}v_2 + \cdots + \overline{u_n}v_n$$

and $<u, av + bw> = a<u,v> + b<u,w>$, but $<cu + dv, w> = \overline{c}<u,w> + \overline{d}<v,w>$

It follows that u in $\mathbb{C}^n$ and v in $\mathbb{C}^m$, if A is an $m \times n$ matrix, then

$$<Au,v> = (Au)'v = (u'A')v = u'(A'v) = <u, A'v>$$

The *usual* or *Euclidean* norm on $\mathbb{R}^n$ or $\mathbb{C}^n$ is defined to be $\|u\| = \sqrt{<u,u>}$

Definition: The set of vectors $\{v_1, v_2, \cdots, v_k\}$ in $\mathbb{R}^n$ or $\mathbb{C}^n$ is said to be *an orthogonal set of vectors* if for $1 \le i < j \le k$, we have $<v_i, v_j> = 0$ and the set is said to be an *orthonormal set of vectors* if it is an orthogonal set and $\|v_i\| = 1$ for all i with $1 \le i \le k$.

$*$ **109.** The Parallelogram Law from Euclidean Geometry is: The sum of the squares of the lengths of the diagonals is equal to the sum of the squares of the lengths of the sides. If u and v are vectors that form the sides of a parallelogram, then the diagonals are $u + v$ and $u - v$. Prove the vector form of the Parallelogram Law

$$\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right)$$

110. Prove that an orthogonal set of non-zero vectors is linearly independent.

$*$ **111.** Let $\mathcal{B} = \{w_1, w_2, \cdots, w_n\}$ be an orthonormal set of vectors in $\mathbb{C}^n$.

(a) Prove that $\mathcal{B}$ is basis for $\mathbb{C}^n$, that is, *an orthonormal basis*, and that for any u in $\mathbb{C}^n$

$$u = <w_1,u>w_1 + <w_2,u>w_2 + \cdots + <w_n,u>w_n$$

(b) Prove: for u and v in $\mathbb{C}^n$, $<u,v> = \sum_{j=1}^{n} \overline{<w_j,u>}\,<w_j,v> = \sum_{j=1}^{n} <u,w_j>\,<w_j,v>$

and therefore that $\|u\|^2 = \sum_{j=1}^{n} |<w_j,u>|^2$

$**$ **112.** Let C and D be $n \times n$ matrices.

(a) Prove that the nullspace of D is a subset of the nullspace of CD.

(b) Prove that the range of CD is a subset of the range of C.

(c) Use the results of (a) and (b) to prove that

$$\text{rank}(CD) \leq \text{rank}(C) \qquad \textit{and} \qquad \text{rank}(CD) \leq \text{rank}(D).$$

$**$ **113.** Let $A = \begin{pmatrix} 1 & 1 & -1 & 2 \\ -1 & 0 & 2 & -3 \\ 1 & -1 & -3 & 4 \end{pmatrix}$

The vectors $v_1 = (2, -1, 1, 0)$ and $v_2 = (-3, 1, 0, 1)$ are a basis for the nullspace of A.

(a) Find a basis for the range of A.

(b) Find a basis for the range of A'.

(c) Find a basis for the orthogonal complement of the range of A'.

(d) Find a basis for the nullspace of A'.