April 12

$*$ **89.** Suppose $\mathcal{V}$ is a vector space over the field F and E and T are, respectively, a projection and a linear transformation on $\mathcal{V}$.

(a) Show that the range of E is invariant for T if and only if $ETE = TE$.

(b) Show that the range and nullspace of E are *both* invariant for T if and only if $TE = ET$.

(c) Which operators commute with *every* projection on $\mathcal{V}$?

$*$ **90.** Suppose $\mathcal{V}$ is a vector space over the field F and for $j = 1, \cdots, k$ the subspaces W_j satisfy

$$\mathcal{V} = W_1 \oplus W_2 \oplus \cdots \oplus W_k$$

Let T be a linear transformation on $\mathcal{V}$ for which the subspaces W_j are invariant for $j = 1, \cdots, k$, let T_j be the restriction of T to W_j, let A_j be the matrix for T_j with respect to the basis $\mathcal{B}_j$ for W_j, and let A be the matrix for T with respect to the basis $\mathcal{B} = \{\mathcal{B}_1, \cdots, \mathcal{B}_k\}$ for $\mathcal{V}$.

(a) Show that $\det(A) = \det(A_1)\det(A_2)\cdots\det(A_k)$.

(b) Prove that if f_j is the characteristic polynomial of T_j and A_j, then the characteristic polynomial of T and A is f, the product of the f_j's.

(c) Prove that the minimal polynomial of T and A is the least common multiple of the minimal polynomials of the T_j's.

$*$ **91.** Let $G = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$

(It might be helpful to observe that 1 is an eigenvalue of G.)

(a) Find the characteristic and minimal polynomials for G and explain how you know that G is diagonalizable over the field $\mathbb{R}$.

(b) Find eigenspaces W_1, W_2, and W_3 that are invariant subspaces for G giving a direct sum decomposition of $\mathbb{R}^4$ as $W_1 \oplus W_2 \oplus W_3$.

(c) Find projections E_1, E_2, and E_3 so that $E_1 + E_2 + E_3 = I$, $E_iE_j = 0$ for $i \neq j$ and $G = aE_1 + bE_2 + cE_3$ for some real numbers a, b, and c.

$*$ **92.** Let T be the linear transformation on $\mathbb{R}^3$ represented in the usual basis by the matrix $\begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix}$

(a) Express the minimal polynomial for T as $p = p_1p_2$ where p_1 and p_2 are monic and irreducible polynomials over $\mathbb{R}$.

(b) For both $j = 1$ and $j = 2$, find a basis $\mathcal{B}_j$ for W_j, the null space of $p_j(T)$.

(c) Find the matrices for T_1 and T_2, the restrictions of T to W_1 and W_2, with respect to these bases, and also find the matrix for T with respect to the basis $\mathcal{B} = \{\mathcal{B}_1, \mathcal{B}_2\}$.

$*$ **93.** Let S be the linear transformation on $\mathbb{R}^3$ represented in the usual basis by the matrix

$$\begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix}$$

Show that there are a diagonalizable operator D and a nilpotent operator N on $\mathbb{R}^3$ so that $S = D + N$ and $DN = ND$. Find the matrices for D and N in the usual basis for $\mathbb{R}^3$.

$*$ **94.** Let T be a linear transformation on a finite dimensional vector space $\mathcal{V}$ that has characteristic polynomial

$$f = (x - c_1)^{d_1}(x - c_2)^{d_2} \cdots (x - c_k)^{d_k}$$

and minimal polynomial

$$p = (x - c_1)^{r_1}(x - c_2)^{r_2} \cdots (x - c_k)^{r_k}$$

Let W_i be the null space of $(T - c_i I)^{r_i}$.

(a) Prove that W_i is an invariant subspace for T.

(b) Letting T_i denote the restriction of T to the invariant subspace W_i, show that $T_i - c_i I$ is nilpotent on W_i and find its order of nilpotence.

(c) Find the minimal polynomial of T_i, the characteristic polynomial of T_i, and the dimension of W_i.