April 5

$**$ **81.** Look back at exercise 75. You chose a vector 'at random' to use for finding the polynomial q that worked for your vector and a given B. Probably the degree of the polynomial you found from using your vector was 4.

Suppose A is 4×4 matrix with complex entries.

(a) For which v in $\mathbb{C}^4$ will A^4v, A^3v, A^2v, Av and Iv be linearly dependent? Why?

(b) For which v in $\mathbb{C}^4$ will Av and Iv be linearly dependent? Why?

(c) For which v in $\mathbb{C}^4$ will A^2v, Av and Iv be linearly dependent? Why?

(d) For which v in $\mathbb{C}^4$ will A^3v, A^2v, Av and Iv be linearly dependent? Why?

(e) Explain why it was extremely likely that choosing a vector 'at random' from R^4 would give a polynomial of degree 4.

82. Let $\mathcal{V}$ be an n-dimensional vector space over the field F. Show that if M is any subspace of $\mathcal{V}$, there is a subspace L of $\mathcal{V}$ for which $M \oplus L = \mathcal{V}$. Indeed, if $\mathcal{V}$ is $\mathbb{R}^n$ or $\mathbb{C}^n$, and $0 < \dim(M) < n$, show that there are infinitely many such subspaces.

$*$ **83.** Let $\mathcal{V}$ be an n-dimensional vector space over the field F and let $W_1, W_2, \cdots, W_k$ be subspaces of $\mathcal{V}$ such that

$$\mathcal{V} = W_1 + W_2 + \cdots + W_k \quad \text{and} \quad \dim(V) = \dim(W_1) + \dim(W_2) + \cdots + \dim(W_k)$$

Prove that this means $\mathcal{V} = W_1 \oplus W_2 \oplus \cdots \oplus W_k$.

$*$ **84.** Let E be an $n \times n$ matrix over the field F such that $E^2 = E$.

(a) Show that $I - E$ is also a projection matrix.

(b) If E is described as the projection onto R along N, what is the description of $I - E$?

(c) Let $Q = \begin{pmatrix} -1 & 2 & -2 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix}$

Show that Q is a projection and describe Q as in part (b).

$*$ **85.** Consider the statement: "If a diagonalizable operator has only eigenvalues 0 and 1, then it is a projection." If it is true, prove it; if it is false, find an example.

$*$ **86.** Let $E_1, E_2, \cdots, E_k$ be projection matrices on $\mathbb{R}^n$ for which $E_1 + E_2 + \cdots + E_k = I$. Use the trace function to show that $E_iE_j = 0$ for $i \neq j$.

$*$ **87.** Let E be a projection on the real vector space $\mathcal{V}$. Prove that $I + E$ is invertible and find $(I + E)^{-1}$.

$*$ **88.** Let P and Q be projections on the real vector space $\mathcal{V}$ for which $PQ = QP$. Prove that PQ is also a projection and find the range and nullspace of PQ.