**March 29**

$*$ **73.** Let $A = \begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix}$

Show that $A$ is not similar, over the field $\mathbb{R}$, to a diagonal matrix. Is $A$ similar to a diagonal matrix over $\mathbb{C}$?

$*$ **74.** Let $B$ be an $n \times n$ matrix over the field $F$ and let $v$ be a vector in $F^n$.

(a) Prove that the set

$$J_v = \{p \in F[x] : p(B)v = 0\}$$

is an ideal in $F[x]$.

(b) Prove that the monic generator, $q$, of $J_v$ must divide the minimal polynomial of $B$ and, therefore, it must divide the characteristic polynomial of $B$.

(c) Conclude that if the degree of $q$ is $n$, then $q$ is actually the characteristic polynomial of $B$.

$*$ **75.**

Let $B = \begin{pmatrix} 1 & 1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & -1 & -1 & 0 \end{pmatrix}$

Choose a non-zero vector $v$ and find a monic polynomial $q$ for which $q(B)v = 0$. Use your answer, $q$, to find the characteristic polynomial for $B$.

$*$ **76.** Let $C$ and $D$ be $n \times n$ matrices over the field $F$.

(a) Prove that if $I - CD$ is invertible, then $I - DC$ is also invertible and

$$(I - DC)^{-1} = I + D(I - CD)^{-1}C$$

(b) Use this result to show that $CD$ and $DC$ have the same eigenvalues over the field $F$.

$*$ **77.** Let $N$ be a linear transformation on an $n$-dimensional vector space $\mathcal{V}$ over the field $F$. Prove: if $N^k = 0$ for some positive integer $k$, then $N^n = 0$.

$*$ **78.** Let $E = \begin{pmatrix} 0 & 1 & 0 \\ 2 & -2 & 2 \\ 2 & -3 & 2 \end{pmatrix}$

Either find an upper triangular matrix, $F$, that is similar to $E$ over the field, $\mathbb{R}$, of real numbers, or prove that $E$ is not similar to any upper triangular matrix over $\mathbb{R}$.

$*$ **79.** Let $G = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$

(a) Suppose $M$ is an invariant subspace for an operator $H$ on a vector space $\mathcal{V}$. Show that the eigenvalues of the restriction of $H$ to $M$ are also eigenvalues of $H$ on $\mathcal{V}$.

(b) Find all the 1-dimensional invariant subspaces for $G$.

(c) Find all the 2-dimensional invariant subspaces for $G$.

$*$ **80.** Find an invertible matrix $S$ so that $S^{-1}PS$ and $S^{-1}QS$ are both diagonal where $P$ and $Q$ are the real matrices

(a) $P = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$ and $Q = \begin{pmatrix} 3 & -8 \\ 0 & -1 \end{pmatrix}$

(b) $P = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}$