Professor Carl Cowen **Math 55400** **Spring 17**

NOTE April 6

Characteristic Polynomials of Matrices

If A is an $n \times n$ matrix over the field F, then the *characteristic polynomial of* A is $\det(xI - A)$. While this is easy to state, actually computing the characteristic polynomial of a matrix from the determinant is not very easy if the matrix is somewhat large, and very difficult if the matrix is actually large. The goal of Exercise 81 is to suggest an alternate strategy for finding the characteristic polynomial for real or complex matrices, highly likely to succeed, but not guaranteed to succeed, but MUCH faster than using determinants.

The experience in our class was that every choice that you made for a non-zero vector in $\mathbb{R}^4$ (no one chose a vector in $\mathbb{C}^4$ that was not in $\mathbb{R}^4$) gave a polynomial of degree 4. Since we know that if p is the characteristic polynomial for A, then $p(A) = 0$, which implies $p(A)v = 0$ for every v in $\mathbb{R}^4$ or $\mathbb{C}^4$. On the other hand, if you choose a vector v and find a polynomial, q, such that $q(A)v = 0$, the polynomial q divides the characteristic polynomial p. In particular, this means that if q is a monic polynomial the degree 4, then $q = p$.

Now, if v is given, to find the polynomial q with the strategy given in problem 81, you need to multiply v by A to get Av, multiply Av by A to get A^2v, multiply A^2v by A to get A^3v, and multiply A^3v by A to get A^4v, and then solve the system of 4 equations in 4 unknowns

$$\alpha A^3v + \beta A^2v + \gamma Av + \delta v = -A^4v$$

to get a polynomial $q(x) = x^4 + \alpha x^3 + \beta x^2 + \gamma x + \delta$ of degree 4 that satisfies $q(A)v = 0$. Note that this system is guaranteed to be consistent, so there are either infinitely many solutions or exactly one. If there are infinitely many, there is one with $\delta = 0$ and then we can divide that version of q by x to get a polynomial of degree 3, etc. (If there is only one solution of degree 4 and $\delta = 0$, then dividing q by x will *not* give a polynomial of the desired type.)

Thus, this strategy can find the characteristic polynomial of a 4×4 matrix by doing four matrix-vector multiplications and solving one system of four equations in four unknowns. Something most of us could do with pencil and paper in a few minutes. Finding the determinant of a 4×4 matrix with symbols in the matrix will take me much longer. Increasing the size to 6×6, I can still do the strategy in exercise 81 but, for determinants are going to be getting out of my computational range. Increasing the size to 100×100 is still in the range of my 1984 MacPlus running Matlab solving the system strategy in less than 10 seconds, but at that time (before parallelism) even a supercomputer would take days or years to do the problem if the determinants were calculated by using expansion by minors!

Thus, the point of exercise 81 is to show, given a 'random' or 'generic' 4×4 matrix A, how likely the strategy of choosing a vector 'at random' from $\mathbb{R}^4$ will work by finding what choice of vectors in $\mathbb{C}^4$ will make the strategy fail. After finding which vectors lead to failure, you can make a judgement of how likely the strategy is to fail.

Addendum to NOTE April 6

Returning to Problem 75, let $B = \begin{pmatrix} 1 & 1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & -1 & -1 & 0 \end{pmatrix}$

In part (c) of Problem 81, if we let A be the matrix B above, then for

$$a = -2.575630959115292 \quad \text{and} \quad b = 2.394455557338821 \quad \text{and} \quad u = \begin{pmatrix} 0.471713928702633 \\ -0.199651595963784 \\ 0.855249114122012 \\ 0.078575839101815 \end{pmatrix}$$

Matlab says the (usual) norm in $\mathbb{R}^4$ of u is 1 and the (usual) norm in $\mathbb{R}^4$ of

$$(B^2 + aB + bI)u$$

is less than 2.23×10^{-15}. What is actually correct is that there are irrational real numbers $\tilde{a}$ and $\tilde{b}$ very close to a and b, respectively, and a vector, $\tilde{u}$, with norm 1 whose components are irrational real numbers and $\tilde{u}$ is very close to u for which

$$(B^2 + \tilde{a}B + \tilde{b}I)\tilde{u} = 0$$

Furthermore, for

$$c = 0.575630959115296 \quad \text{and} \quad d = 2.088157361983758 \quad \text{and} \quad w = \begin{pmatrix} -0.402203489404201 \\ 0.795028832339588 \\ 0.115232096856176 \\ 0.439184554275266 \end{pmatrix}$$

Matlab says the usual norm of $(B^2 + cB + dI)w$ is less than 2.38×10^{-15}.

This means there are at least two non-zero vectors v with real coordinates that have B^2v, Bv and v linearly dependent in $\mathbb{R}^4$.