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Math 53000 **Test II** **April 10, 2014**

There are 5 questions, 5 pages, and 100 points on this test.
No calculators, No books, No notes, Ask for scrap paper if you need it, $\cdots$ Test ends at 4:20p.

(20 points) 1. Let f and g be holomorphic and non-constant on the punctured disk $0 < |z - z_0| < 1$ for some z_0 in $\mathbb{C}$. Suppose, in addition, that $\lim_{z\to z_0} f(z) = \lim_{z\to z_0} g(z) = 0$.

(a) What kind of singularities do f and g have at z_0? Explain how you know.

(b) Prove that if either $\lim_{z\to z_0} f'(z) \neq 0$ *OR* $\lim_{z\to z_0} g'(z) \neq 0$, then

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \lim_{z\to z_0} \frac{f'(z)}{g'(z)}$$

(c) Prove there is a positive integer k, $\lim_{z\to z_0} f^{(k)}(z) \neq 0$, where $f^{(k)}$ denotes the k^{th} derivative of f.

2. For n a positive integer, let a_1, a_2, $\cdots$, a_n be points on the unit circle (that is, the circle of radius 1 with center at the origin) and let p be the polynomial $p(z) = \prod_{k=1}^{n}(z - a_k)$.

(a) Find $|p(0)|$.

(b) Prove that there is a point z_0 on the unit circle such that $|p(z_0)| = 1$.

(20 points) 3. Evaluate:

$$\int_{-\infty}^{\infty} \frac{x^2 - 1}{1 + x^4}\, dx$$

(20 points) 4. The function h is holomorphic in the entire plane, except for a singularity at $z = 0$. There are M_1 and M_2, positive numbers, and a positive integer n

such that $|h(z)| \leq \dfrac{M_1}{|z|^n}$ for $0 < |z| < 2$

and $|h(z)| \leq M_2$ for $1 < |z| < \infty$.

(a) Give as complete a description as you can, as well as giving reasons for the parts of the description, for the Laurent series for h centered at 0.

(b) Use this description to classify the possible kinds of singularity h has at 0.

(20 points) 5. Evaluate the integral $\displaystyle\int_0^{\infty} \frac{1}{(1 + x)\sqrt{x}}\, dx$

by integrating a suitable complex valued function along the piecewise smooth curve, oriented counter-clockwise, surrounding the region consisting of the annulus $\{z : 1/R \leq |z| \leq R\}$ with the part of the annulus in the rectangle $\{z = x + iy : 0 < x < R \text{ and } -1/R^2 < y < 1/R^2\}$ removed (see the diagram).

In your write up, you may skip the details such as showing that the integral along the line segment just above the x-axis converges properly to the integral along the axis, and the integral on the circle near the origin differs from the integral of $1/\sqrt{z}$ along the circle by an error converging to zero.