NAME: ____________________

Math 53000 **Final Exam** **May 8, 2014**

There are 6 questions, 6 pages, and 150 points on this test.
No calculators, No books, No notes, Ask for scrap paper if you need it, $\cdots$ Exam ends at 5:30p.

Notation: unit disk: $\mathbb{D} = \{z : |z| < 1\}$ and upper half plane: $H_+ = \{z : \mathrm{Im}(z) > 0\}$

(25 points) 1. Find all linear fractional maps, φ, that map the upper half plane H_+ *onto* itself and satisfy $\lim_{z\to\infty} \varphi(z) = \infty$. Provide a step by step justification for your answer!

Are there any φ *as above* such that $\varphi(3+i) = -2+4i$? If so, find all such φ.

(25 points) 2. Suppose g is an analytic function on $\mathbb{D}$ and $g(\mathbb{D}) \subset \mathbb{D}$.

Prove: *If a and b, with $a \neq b$, are points of $\mathbb{D}$ such that $g(a) = a$ and $g(b) = b$, then $g(z) = z$ for every z in the disk.*

(25 points) 3. Evaluate $\displaystyle\int_0^\infty \frac{\sqrt{x}}{1+x^2}\,dx$ by using contour integration on boundary of the set:

$$\{z = re^{i\theta} : \epsilon < r < R \text{ and } 0 < \theta < \pi\}$$

(25 points) 4. The function f is holomorphic on the connected open set Ω in the complex plane and there is no connected open set $\Omega_1 \supset \Omega$ with $\Omega_1 \neq \Omega$ on which there is g holomorphic on Ω_1 and $g(z) = f(z)$ for z in Ω. In other words, Ω is a maximal domain on which f is holomorphic.

For those z in Ω for which the power series converges, $\displaystyle f(z) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}(z+1+i)^n$

(a) What is the largest open set on which this power series converges?

(b) Does Ω contain the set on which the power series converges, that is, is f holomorphic on the set on which the power series converges? If so, cite some theorem that shows that is the case. If not, explain why not.

(c) Does f' have a power series on some part of Ω? If so, cite some theorem that shows that is the case and, if you can, find such a power series for f'. If not, explain why not.

(d) What is the largest set on which you can be sure f is holomorphic, and explain your answer.

(25 points) 5. Let $\sqrt{\cdot}$ denote the branch of the square root function defined on $\mathbb{C} \setminus (-\infty, 0]$ and satisfying $\sqrt{4} = 2$.

Let $\varphi(z) = \dfrac{1+z}{1-z}$ and, finally, let $f(z) = \dfrac{\sqrt{\varphi(z)} - 1}{\sqrt{\varphi(z)} + 1}$

Note: It is true, and you do NOT need to prove it, that f is holomorphic on the unit disk, $\mathbb{D}$, that f has a continuous extension to the closed disk, and that f is one-to-one on the closed disk.

"Estimate" means you do not have to justify your answer!!

(a) Describe the set $\Omega = f(\mathbb{D})$.

(b) For $r > 0$ and ζ, with $|\zeta| = 1$, let $\Delta(\zeta, r) = \{w : |w - f(\zeta)| < r\}$, the disk of radius r, center $f(\zeta)$.

In order to highlight some features of Ω, for each ζ with $|\zeta| = 1$, estimate $\displaystyle\lim_{r\to 0^+} \frac{\mathrm{area}(\Omega \cap \Delta(\zeta, r))}{\mathrm{area}(\Delta(\zeta, r))}$

(25 points) 6. For each positive integer n, let P_n be the polynomial $P_n(z) = 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \cdots + \dfrac{z^n}{n!}$

Prove that for any $R > 0$,

there is N so that $n > N$ implies the polynomial P_n has exactly n zeros in $\{z : |z| > R\}$.