Exercises 1.2

1. Let $u = (1, -1, 3)$, $v = (0, 2, -1)$, and $w = (3, 1, 1)$. Evaluate the following expressions:

   (a) $4u$ (b) $-3v$ (c) $u + w$ (d) $4u - 3v$ (e) $2u - 4v + 3w$

2. Let $u = (2, 1, 0, -3)$, $v = (1, 0, 3, -1)$, $w = (2, 0, 6, -2)$, and $x = (1, -2, 1)$. Evaluate the following expressions when possible; say *Undefined* when the arithmetic in the expression cannot be carried out.

   (a) $3u - 2v$ (b) $2u + v - 3w$ (c) $3x + w$ (d) $\alpha u + \beta v + \gamma w$

3. Let u, v, and w be vectors as in the previous problem.
   (a) Find α and β so that $\alpha u + \beta v = (1, 2, -9, -3)$.
   (b) Find γ and δ so that $\gamma u + \delta w = (3, -1, 2, 0)$.
   (c) Find ϵ and ζ so that $\epsilon v + \zeta w = (-1, 0, -3, 1)$.

4. Let $A = \begin{pmatrix} 5 & -4 & 1 \\ 12 & -11 & 6 \\ 10 & -10 & 8 \end{pmatrix}$
   and $u = (1, -1, 2)$, $v = (1, 1, 0)$, $w = (1, 2, 1)$, $e_1 = (1, 0, 0)$, and $e_2 = (0, 1, 0)$.
   (a) Find Au.
   (b) Find Av.
   (c) Find Aw.
   (d) Find Ae_1 and Ae_2. Let $e_3 = (0, 0, 1)$; guess what Ae_3 is, then compute it.

5. Let $M = \begin{pmatrix} 0 & 2 & -1 \\ 3 & 1 & 1 \end{pmatrix}$ and $N = \begin{pmatrix} -1 & 0 & 4 \\ 1 & 1 & -2 \end{pmatrix}$. Evaluate the following expressions.

   (a) $3M$ (b) $-2N$ (c) $M + N$ (d) $3M - 2N$ (e) M'
   (f) N' (g) $(3M - 2N)'$ (h) MM' (i) $M'M$ (j) MN'

6. Let $S = \begin{pmatrix} 0 & 2 & -1 \\ 2 & 1 & 3 \\ -1 & 3 & -2 \end{pmatrix}$.
   (a) Find S'.
   (b) What special property does S have?
   (c) What is $S + I$. How do you know which identity matrix to add to S?
   (d) Find S^2 and S^3.
   (e) Show that if T is any Hermitian matrix, then T^2 is Hermitian also.

7. Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 0 & 3 \end{pmatrix}$, $C = \begin{pmatrix} 1 & -3 \\ -1 & 3 \\ -1 & 3 \end{pmatrix}$,
   $D = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, and $E = \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$. Evaluate the following expressions when possible; say *Undefined* when the arithmetic in the expression cannot be carried out.

   (a) $3A - 2B$ (b) AE (c) AB (d) AC (e) CA
   (f) EA (g) $E'A$ (h) $AB' + D$ (i) A^2 (j) D^2

8. Let $P = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, $Q = \begin{pmatrix} 2 & -1 & 1 \\ -3 & 4 & -2 \\ 5 & 3 & -5 \end{pmatrix}$, and let D be the diagonal matrix $D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.
   (a) Find DP and DQ.
   (b) If E is the diagonal matrix with diagonal entries α, β, and γ, and R is a matrix, describe ER.
   (c) Find PD and QD.
   (d) If E is the diagonal matrix with diagonal entries α, β, and γ, and R is a matrix, describe RE.

9. Let $S = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$ and let $T = \begin{pmatrix} 2 & 1 & 3 \\ 1 & -1 & -2 \end{pmatrix}$.
   Let $C_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, let $C_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, and let $C_3 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$.
   (a) Find SC_1, SC_2, and SC_3.
   (b) Find ST and compare your answer with the results of part a).

10. (a) Let $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 2 \\ 2 & -1 & 1 \end{pmatrix}$ and let $B = \begin{pmatrix} -4 & -1 & 2 \\ -5 & -1 & 2 \\ 3 & 1 & -1 \end{pmatrix}$.
   Explain why $A = B^{-1}$.
   (b) Is $B = A^{-1}$? Explain!
   (c) Let $C = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$, and $D = \begin{pmatrix} -2 & 1 \\ -3 & 1 \\ 1 & 1 \end{pmatrix}$. Is $D = C^{-1}$? Explain!

11. Let $E = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$. Find a matrix $F = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ so that $F = E^{-1}$.

12. (a) Show that if G is an invertible matrix, then G' is also invertible and $(G')^{-1} = (G^{-1})'$.
   (b) Use your answer to part (a) and problem 10 above to find the inverse of $\begin{pmatrix} 1 & -1 & 2 \\ -1 & 2 & -1 \\ 0 & 2 & 1 \end{pmatrix}$.

13. Verify that if N is a matrix such that $N^4 = 0$, then
$$(I - N)^{-1} = I + N + N^2 + N^3.$$
   WARNING! Such matrices are called *nilpotent* and are **not** necessarily 0.
   For example, the matrix $M = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ satisfies $M^2 = 0$.

14. Let E be an $m \times n$ matrix.
   (a) Show that EE' and $E'E$ are both Hermitian.
   (b) Give an example to show that these are not always the same.
   (c) Show that if E is square, then $E + E'$ is Hermitian.

15. Redo Exercises 7, 10, and 11 using a suitable machine. How does your machine react to undefined matrix operations?

16.
$$F = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & -1 & 4 & 3 \\ 4 & 2.6 & 0 & 3 \\ 3 & -.3 & 8 & 1.5 \end{pmatrix}$$
   (a) Use a suitable machine to find $G = F^{-1}$
   (b) Find the computed values of GF and $GF - I$. Explain the output of your machine.

Exercises 1.3

1. Let $A = \begin{pmatrix} 4 & 3 & -2 \\ 2 & -5 & 6 \end{pmatrix}$ and let $B = \begin{pmatrix} 0 & -1 & 3 \\ 2 & -1 & 6 \\ 5 & 2 & 1 \end{pmatrix}$.
   (a) Find AB (from the definition) as a 2×3 matrix.
   (b) Partition A as $\left(A_{11} \mid A_{12} \right)$ and B as $\left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array} \right)$, where both A_{11} and B_{11} are 2×2 matrices, that is, say what each of $A_{11}, A_{12}, \cdots, B_{22}$ are.
   (c) Determine each of the relevant products from (b) above and find AB as a partitioned matrix.
2. Use partitioned matrices to show that if A is an $m \times n$ matrix and B is an $n \times p$ matrix whose k^{th} column is zero, then the k^{th} column of AB is zero.
3. Explore how your software handles block matrices.
   (a) Enter the matrices

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 4 & 2.6 & 0 \\ 3 & -.3 & 8 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & -2 \\ 1.5 & 4 \end{pmatrix}$$

   (b) Make a 4×5 matrix E from the matrices A, B, C, and D to get

$$E = \left(\begin{array}{ccc|cc} 1 & 2 & -1 & 1 & -1 \\ 0 & -1 & 4 & 3 & 1 \\ \hline 4 & 2.6 & 0 & 3 & -2 \\ 3 & -.3 & 8 & 1.5 & 4 \end{array} \right)$$

   You probably do not need to retype all the entries! Note that $E = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

   (c) Make a 4×4 matrix F from E by deleting its last column $F = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & -1 & 4 & 3 \\ 4 & 2.6 & 0 & 3 \\ 3 & -.3 & 8 & 1.5 \end{pmatrix}$
4. Prove: If a matrix is multiplied on the right by a diagonal matrix, the j^{th} column of the product is the j^{th} diagonal entry times the the j^{th} column of the original matrix. (Compare with Exercise 8)
5. Suppose A is a square matrix partitioned as

$$A = \left(\begin{array}{c|c} X & Y \\ \hline 0 & Z \end{array} \right)$$

   where X and Z are square invertible matrices and 0 is a zero matrix.
   (a) Find formulas for P, Q, R, and S so that the block matrix

$$\left(\begin{array}{c|c} P & Q \\ \hline R & S \end{array} \right)$$

   is A^{-1}. (*Caution:* matrix multiplication is not commutative!) If you are successful, you will have shown that matrices with the given block form are invertible.
   (b) Use your formula to find A^{-1} when $X = \left(-1 \right)$, $Y = \left(1 \quad -1 \right)$, and $Z = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$ (Note that $Z^{-1} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$)

Exercises 2.1

1. $\begin{cases} w & = 5 \\ 2w + x & = 2 \\ w + x + y & = -1 \\ w - x + 2y + z & = 4 \end{cases}$

2. $\begin{cases} a + 2b + c & = 2 \\ -a + 3b - c + d & = 3 \\ a & = 4 \\ 2a + b & = -1 \end{cases}$

3. $\begin{cases} 2w - x + 2y + z & = 1 \\ x + y - z & = -2 \\ 3y + z & = 0 \\ 2z & = 6 \end{cases}$

4. $\begin{cases} 2w - x + 2y + z & = 0 \\ x + y - z & = 0 \\ y - z & = 0 \end{cases}$

(Hint: solve for w, x, and y in terms of z. There will be infinitely many solutions, one for each value of z.)

5. Write each system in Problems 6–9 as a matrix equation.

Use your software to solve the following systems. Be sure to check your answers!

6. $\begin{cases} x + 2y & = 3 \\ 3x + 4y & = -2 \end{cases}$

7. $\begin{cases} x - y + z & = 1 \\ -x + 3y + 3z & = 5 \\ 2x + 3z & = 4 \end{cases}$

8. $\begin{cases} w - y + 2z & = 0 \\ -w + x + 3y - z & = 5 \\ 2w + 5z & = 3 \\ w + x + y + 2z & = 4 \end{cases}$

9. $\begin{cases} 2w + 3x + y - z & = 1 \\ -w + 2x + 3y + z & = -1 \\ 2w + x - 2y + 3z & = 0 \\ w - x + y + 2z & = 2 \end{cases}$

10. The five-tuples $(2, 2, 1, -1, 1)$ and $(1, 1, 2, -1, -1)$ are both solutions of the system:

$$\begin{cases} a + b + 4c + d + e & = 8 \\ a - b + 2c + 2d + e & = 1 \\ 2a + b - c - d - 2e & = 4 \\ b + 3c + d + e & = 5 \\ 2a - b + c + 3d & = 0 \end{cases}$$

(a) Without using Gaussian elimination or a machine, write down two non-trivial solutions of the associated homogeneous system.
(b) Write down two other solutions of the given system.

11. Let A be the matrix

$$\begin{pmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & -3 & -1 \\ 1 & 1 & 3 & -2 \\ -1 & 2 & -2 & 3 \end{pmatrix}$$

and let $b = (3, -1, 3, 2)$ and let $c = (0, 4, -4, 4)$.
(a) Check that $Y = (1, 1, 1, 1)$ solves the system $AX = b$ and that $Z = (1, 0, -1, 1)$ solves the system $AX = c$.
(b) Without using Gaussian Elimination or a machine, find a solution of the system $AX = (6, -2, 6, 4) = 2b$.
(c) Without using Gaussian Elimination or a machine, find a solution of the system $AX = (3, 3, -1, 6) = b + c$.
(d) Without using Gaussian Elimination or a machine, find a solution of the system $AX = (9, 5, 1, 14) = 3b + 2c$.