Exercises 1.3

1. Let $A = \begin{pmatrix} 4 & 3 & -2 \\ 2 & -5 & 6 \end{pmatrix}$ and let $B = \begin{pmatrix} 0 & -1 & 3 \\ 2 & -1 & 6 \\ 5 & 2 & 1 \end{pmatrix}$.
   (a) Find AB (from the definition) as a 2×3 matrix.
   (b) Partition A as $\left(A_{11} \mid A_{12} \right)$ and B as $\left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right)$, where both A_{11} and B_{11} are 2×2 matrices, that is, say what each of $A_{11}, A_{12}, \cdots, B_{22}$ are.
   (c) Determine each of the relevant products from (b) above and find AB as a partitioned matrix.
2. Use partitioned matrices to show that if A is an $m \times n$ matrix and B is an $n \times p$ matrix whose k^{th} column is zero, then the k^{th} column of AB is zero.
3. Explore how your software handles block matrices.
   (a) Enter the matrices

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 4 & 2.6 & 0 \\ 3 & -.3 & 8 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & -2 \\ 1.5 & 4 \end{pmatrix}$$

   (b) Make a 4×5 matrix E from the matrices A, B, C, and D to get

$$E = \left(\begin{array}{ccc|cc} 1 & 2 & -1 & 1 & -1 \\ 0 & -1 & 4 & 3 & 1 \\ \hline 4 & 2.6 & 0 & 3 & -2 \\ 3 & -.3 & 8 & 1.5 & 4 \end{array}\right)$$

   You probably do not need to retype all the entries! Note that $E = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

   (c) Make a 4×4 matrix F from E by deleting its last column $F = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & -1 & 4 & 3 \\ 4 & 2.6 & 0 & 3 \\ 3 & -.3 & 8 & 1.5 \end{pmatrix}$
4. Prove: If a matrix is multiplied on the right by a diagonal matrix, the j^{th} column of the product is the j^{th} diagonal entry times the the j^{th} column of the original matrix. (Compare with Exercise **??**)
5. Suppose A is a square matrix partitioned as

$$A = \left(\begin{array}{c|c} X & Y \\ \hline 0 & Z \end{array}\right)$$

   where X and Z are square invertible matrices and 0 is a zero matrix.
   (a) Find formulas for P, Q, R, and S so that the block matrix

$$\left(\begin{array}{c|c} P & Q \\ \hline R & S \end{array}\right)$$

   is A^{-1}. (*Caution:* matrix multiplication is not commutative!) If you are successful, you will have shown that matrices with the given block form are invertible.
   (b) Use your formula to find A^{-1} when $X = \begin{pmatrix} -1 \end{pmatrix}$, $Y = \begin{pmatrix} 1 & -1 \end{pmatrix}$, and $Z = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$ (Note that $Z^{-1} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$)

Exercises 2.1

1. $\begin{cases} w &= 5 \\ 2w + x &= 2 \\ w + x + y &= -1 \\ w - x + 2y + z &= 4 \end{cases}$

2. $\begin{cases} a + 2b + c &= 2 \\ -a + 3b - c + d &= 3 \\ a &= 4 \\ 2a + b &= -1 \end{cases}$

3. $\begin{cases} 2w - x + 2y + z &= 1 \\ x + y - z &= -2 \\ 3y + z &= 0 \\ 2z &= 6 \end{cases}$

4. $\begin{cases} 2w - x + 2y + z &= 0 \\ x + y - z &= 0 \\ y - z &= 0 \end{cases}$

   (Hint: solve for w, x, and y in terms of z. There will be infinitely many solutions, one for each value of z.)

5. Write each system in Problems 6–9 as a matrix equation.

Use your software to solve the following systems. Be sure to check your answers!

6. $\begin{cases} x + 2y &= 3 \\ 3x + 4y &= -2 \end{cases}$

7. $\begin{cases} x - y + z &= 1 \\ -x + 3y + 3z &= 5 \\ 2x + 3z &= 4 \end{cases}$

8. $\begin{cases} w - y + 2z &= 0 \\ -w + x + 3y - z &= 5 \\ 2w + 5z &= 3 \\ w + x + y + 2z &= 4 \end{cases}$

9. $\begin{cases} 2w + 3x + y - z &= 1 \\ -w + 2x + 3y + z &= -1 \\ 2w + x - 2y + 3z &= 0 \\ w - x + y + 2z &= 2 \end{cases}$

10. The five-tuples $(2, 2, 1, -1, 1)$ and $(1, 1, 2, -1, -1)$ are both solutions of the system:

$$\begin{cases} a + b + 4c + d + e &= 8 \\ a - b + 2c + 2d + e &= 1 \\ 2a + b - c - d - 2e &= 4 \\ b + 3c + d + e &= 5 \\ 2a - b + c + 3d &= 0 \end{cases}$$

   (a) Without using Gaussian elimination or a machine, write down two non-trivial solutions of the associated homogeneous system.
   (b) Write down two other solutions of the given system.

11. Let A be the matrix

$$\begin{pmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & -3 & -1 \\ 1 & 1 & 3 & -2 \\ -1 & 2 & -2 & 3 \end{pmatrix}$$

   and let $b = (3, -1, 3, 2)$ and let $c = (0, 4, -4, 4)$.
   (a) Check that $Y = (1, 1, 1, 1)$ solves the system $AX = b$ and that $Z = (1, 0, -1, 1)$ solves the system $AX = c$.
   (b) Without using Gaussian Elimination or a machine, find a solution of the system $AX = (6, -2, 6, 4) = 2b$.
   (c) Without using Gaussian Elimination or a machine, find a solution of the system $AX = (3, 3, -1, 6) = b + c$.
   (d) Without using Gaussian Elimination or a machine, find a solution of the system $AX = (9, 5, 1, 14) = 3b + 2c$.