Exercises 1.2

1. Let $u = (1, -1, 3)$, $v = (0, 2, -1)$, and $w = (3, 1, 1)$. Evaluate the following expressions:
   (a) $4u$ (b) $-3v$ (c) $u + w$ (d) $4u - 3v$ (e) $2u - 4v + 3w$

2. Let $u = (2, 1, 0, -3)$, $v = (1, 0, 3, -1)$, $w = (2, 0, 6, -2)$, and $x = (1, -2, 1)$. Evaluate the following expressions when possible; say *Undefined* when the arithmetic in the expression cannot be carried out.
   (a) $3u - 2v$ (b) $2u + v - 3w$ (c) $3x + w$ (d) $\alpha u + \beta v + \gamma w$

3. Let u, v, and w be vectors as in the previous problem.
   (a) Find α and β so that $\alpha u + \beta v = (1, 2, -9, -3)$.
   (b) Find γ and δ so that $\gamma u + \delta w = (3, -1, 2, 0)$.
   (c) Find ϵ and ζ so that $\epsilon v + \zeta w = (-1, 0, -3, 1)$.

4. Let $A = \begin{pmatrix} 5 & -4 & 1 \\ 12 & -11 & 6 \\ 10 & -10 & 8 \end{pmatrix}$
   and $u = (1, -1, 2)$, $v = (1, 1, 0)$, $w = (1, 2, 1)$, $e_1 = (1, 0, 0)$, and $e_2 = (0, 1, 0)$.
   (a) Find Au.
   (b) Find Av.
   (c) Find Aw.
   (d) Find Ae_1 and Ae_2. Let $e_3 = (0, 0, 1)$; guess what Ae_3 is, then compute it.

5. Let $M = \begin{pmatrix} 0 & 2 & -1 \\ 3 & 1 & 1 \end{pmatrix}$ and $N = \begin{pmatrix} -1 & 0 & 4 \\ 1 & 1 & -2 \end{pmatrix}$. Evaluate the following expressions.
   (a) $3M$ (b) $-2N$ (c) $M + N$ (d) $3M - 2N$ (e) M'
   (f) N' (g) $(3M - 2N)'$ (h) MM' (i) $M'M$ (j) MN'

6. Let $S = \begin{pmatrix} 0 & 2 & -1 \\ 2 & 1 & 3 \\ -1 & 3 & -2 \end{pmatrix}$.
   (a) Find S'.
   (b) What special property does S have?
   (c) What is $S + I$. How do you know which identity matrix to add to S?
   (d) Find S^2 and S^3.
   (e) Show that if T is any Hermitian matrix, then T^2 is Hermitian also.

7. Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 0 & 3 \end{pmatrix}$, $C = \begin{pmatrix} 1 & -3 \\ -1 & 3 \\ -1 & 3 \end{pmatrix}$,
   $D = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, and $E = \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$. Evaluate the following expressions when possible; say *Undefined* when the arithmetic in the expression cannot be carried out.
   (a) $3A - 2B$ (b) AE (c) AB (d) AC (e) CA
   (f) EA (g) $E'A$ (h) $AB' + D$ (i) A^2 (j) D^2

8. Let $P = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, $Q = \begin{pmatrix} 2 & -1 & 1 \\ -3 & 4 & -2 \\ 5 & 3 & -5 \end{pmatrix}$, and let D be the diagonal matrix $D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.
   (a) Find DP and DQ.

(b) If E is the diagonal matrix with diagonal entries α, β, and γ, and R is a matrix, describe ER.
(c) Find PD and QD.
(d) If E is the diagonal matrix with diagonal entries α, β, and γ, and R is a matrix, describe RE.

9. Let $S = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$ and let $T = \begin{pmatrix} 2 & 1 & 3 \\ 1 & -1 & -2 \end{pmatrix}$.
Let $C_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, let $C_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, and let $C_3 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$.
(a) Find SC_1, SC_2, and SC_3.

(b) Find ST and compare your answer with the results of part a).

10. (a) Let $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 2 \\ 2 & -1 & 1 \end{pmatrix}$ and let $B = \begin{pmatrix} -4 & -1 & 2 \\ -5 & -1 & 2 \\ 3 & 1 & -1 \end{pmatrix}$.
Explain why $A = B^{-1}$.
(b) Is $B = A^{-1}$? Explain!
(c) Let $C = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$, and $D = \begin{pmatrix} -2 & 1 \\ -3 & 1 \\ 1 & 1 \end{pmatrix}$. Is $D = C^{-1}$? Explain!

11. Let $E = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$. Find a matrix $F = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ so that $F = E^{-1}$.

12. (a) Show that if G is an invertible matrix, then G' is also invertible and $(G')^{-1} = (G^{-1})'$.
(b) Use your answer to part (a) and problem 10 above to find the inverse of $\begin{pmatrix} 1 & -1 & 2 \\ -1 & 2 & -1 \\ 0 & 2 & 1 \end{pmatrix}$.

13. Verify that if N is a matrix such that $N^4 = 0$, then
$$(I - N)^{-1} = I + N + N^2 + N^3.$$
WARNING! Such matrices are called *nilpotent* and are **not** necessarily 0.
For example, the matrix $M = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ satisfies $M^2 = 0$.

14. Let E be an $m \times n$ matrix.
(a) Show that EE' and $E'E$ are both Hermitian.
(b) Give an example to show that these are not always the same.
(c) Show that if E is square, then $E + E'$ is Hermitian.

15. Redo Exercises 7, 10, and 11 using a suitable machine. How does your machine react to undefined matrix operations?

16.
$$F = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & -1 & 4 & 3 \\ 4 & 2.6 & 0 & 3 \\ 3 & -.3 & 8 & 1.5 \end{pmatrix}$$
(a) Use a suitable machine to find $G = F^{-1}$
(b) Find the computed values of GF and $GF - I$. Explain the output of your machine.