Constrained Maxima and Minima for Real-valued Functions of Several Variables

Let f be a real valued function defined and smooth on an open set U in $\mathbb{R}^n$ and let K be a compact set such that $K \subset U$. We say *the real number b is the maximum value of f on K and the maximum value occurs at x_0* if

(a) x_0 is a point in K

(b) $f(x) \leq b$ for every x in K

and

(c) $f(x_0) = b$.

Similarly, we say the real number s is the minimum value of f on K and it occurs at x_1 if x_1 is in K, $f(x) \geq s$ for every x in K, and $f(x_1) = s$. The problem of finding the maximum and minimum values of f on K is called a problem of *constrained extrema.* The ideas of local (or relative) maxima and minima can be extended to this situation: we say *f has a local maximum value on K at x_0* if there is an open subset V of U such that x_0 is in V and $f(x) \leq f(x_0)$ for every x in $K \cap V$.

Very often, the set K is a manifold, the union of several manifolds, or the union of a manifold or several manifolds with an open set. In order find the maximum and minimum values on such a set K, one should find the local maxima on each of the pieces and then choose the largest to get the maximum on K, and similarly to find minimum, find the local minima on each of the pieces and choose the smallest.

If f is defined on an open set $\mathbb{R}^n$, the local maxima and local minima of f occur at the critical points of f. This means each point, x_0, for which $Df = 0$ is a potential local maximum or minimum of f.

We have seen that for a real-valued function, f, on an open set U in $\mathbb{R}^n$, the gradient of f at each point of U is a vector pointing in the direction of the largest increase of the function f. This means that, at each point, the gradient is perpendicular to the level curve of f at that point. Suppose M is an $(n-1)$-dimensional manifold in $\mathbb{R}^n$ determined by the smooth function g, that is, suppose $M = \{x \in \mathbb{R}^n : g(x) = 0\}$. Since M is a level set of g, then ∇g is perpendicular to M at each point of M. If x is a point on manifold M and $\nabla f(x)$ is not perpendicular to M, then moving along M more or less in the direction of $\nabla f(x)$ will result in an increase of f and moving in the opposite direction along M results in a decrease. Thus, at the local extrema of f on M the vectors ∇f and ∇g are both perpendicular to M and we have the following.

Theorem on Lagrange Multipliers:

Suppose M is an $(n-1)$-dimensional manifold in $\mathbb{R}^n$ determined by the smooth function g, that is, suppose $M = \{x \in \mathbb{R}^n : g(x) = 0\}$. If f is a real valued function defined in an open set U such that $U \supset M$ and f has a local maximum or minimum at the point x_0 on M, then either $\nabla f(x_0) = 0$, or else there is a number λ such that $\lambda \nabla f(x_0) = \nabla g(x_0)$.

Example. Let $\mathbf{D} = \{(x,y) \in \mathbb{R}^2 : x^2 - 2x + y^2 \leq 12\}$, which is the disk of radius $\sqrt{13}$ and center $(1,0)$ in the plane. Find the maximum and minimum values of the function $f(x,y) = 2x + 3y + 5$ on the (compact) set $\mathbf{D}$ and find the points of $\mathbf{D}$ at which these values occur.

Solution. Since the compact set $\mathbf{D}$ includes the open subset $\mathbf{D}^\circ = \{(x,y) \in \mathbb{R}^2 : x^2 - 2x + y^2 < 12\}$, we will break the problem into two parts. First, find points of $\mathbf{D}^\circ$ at which f may have a local maximum or minimum, which will be candidates for points at which the maximum and minimum values of f on $\mathbf{D}$ occur. Then, we will find the points of the smooth manifold

$$\partial\mathbf{D} = \{(x,y) \in \mathbb{R}^2 : x^2 - 2x + y^2 = 12\}$$

at which the maximum and minimum values of f might occur.

We compute Df and set it equal to zero to find the critical points of f:

$$Df(x,y) = \begin{pmatrix} 2 & 3 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \end{pmatrix}$$

for any (x,y), so f has no critical points and therefore there are no local maxima or minima of f in $\mathbf{D}^\circ$.

To locate the local maxima and minima of f on $\partial\mathbf{D}$, we will use Lagrange multipliers with f and the function $g(x,y) = x^2 - 2x + y^2 - 12$ because $\partial\mathbf{D}$ is the smooth manifold determined by $g = 0$. In the language of gradients, we want to find points of $\partial\mathbf{D}$ for with $\lambda\nabla f = \nabla g$. We have

$$\nabla f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \nabla g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x-2 \\ 2y \end{pmatrix}$$

so we must solve the system

$$\begin{cases} \lambda \cdot 2 &= 2x - 2 \\ \lambda \cdot 3 &= 2y \\ x^2 - 2x + y^2 - 12 &= 0 \end{cases}$$

Since we are interested in x and y and not λ, really, and because it is not inconvenient to do so, we can use the first two equations to eliminate λ and get a relationship between x and y for the critical points. The first equation gives $\lambda = x - 1$ and using this in the second equation, we find $3(x-1) = 2y$ or $y = (3/2)(x-1)$. Putting this into the last equation, we get

$$\begin{aligned} x^2 - 2x + \frac{9}{4}(x-1)^2 - 12 &= 0 \\ \frac{13}{4}x^2 - \frac{26}{4}x + \frac{9}{4} - 12 &= 0 \\ 13x^2 - 26x - 39 &= 0 \\ x^2 - 2x - 3 &= 0 \\ (x-3)(x+1) &= 0 \end{aligned}$$

so the possible maxima and minima are at $x = 3$ (which means $y = (3/2)(3-1) = 3$) and $x = -1$ (which means $y = (3/2)(-1-1) = -3$). We also find that $\lambda = x - 1$, or $\lambda = 2$ for the point $(3,3)$ and $\lambda = -2$ for the point $(-1,-3)$, although this is not especially interesting. We should check that the two points are really on $\partial\mathbf{D}$ and easy calculations show that they both are.

Finally, we find that $f(3,3) = 20$ and $f(-1,-3) = -6$, so we find that the maximum value of f on $\mathbf{D}$ is 20 and it occurs at $(3,3)$ and that the minimum value of f is -6 and it is achieved at $(-1,-3)$.

MORE TO BE ADDED SOON!