## Some Basic Curves and Surfaces

**Lines** A *line* can be described in two basic ways, as the intersection of two planes or in a parametric form. Both of these ways were described in Math 261, or other multivariable calculus course you have had, but this is a review of the parametric form. If $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$ are two points in $\mathbb{R}^3$, then a parametric representation of the line passing through them is

$$\gamma(t) = \begin{pmatrix} tp_1 + (1-t)q_1 \\ tp_2 + (1-t)q_2 \\ tp_3 + (1-t)q_3 \end{pmatrix} = q + t(p-q) \quad \text{for} \quad -\infty < t < \infty$$

Notice that $0 \le t \le 1$ parametrizes the part of the line that is the line segment with end points $q$ and $p$.

**Cones** A *cone* is a set in which there is a distinguished point in the set called the *vertex* and each other point of the set is contained in the line passing through that point and the vertex, and the set contains this entire line. For example, the set $\mathbf{K}$ parametrized by

$$\gamma\begin{pmatrix} t \\ \theta \end{pmatrix} = \begin{pmatrix} t\cos\theta \\ t\sin\theta \\ t \end{pmatrix} \quad \text{for} \quad -\infty < t < \infty \quad \text{and} \quad 0 \le \theta < 2\pi$$

is a cone. In particular, because $\mathbf{K}$ contains each of the circles $(t\cos(\theta), t\sin(\theta), t)$ which lie in the parallel planes $z = t$ and the line joining their centers (in this case, the $z$-axis) is perpendicular to each of these planes, $\mathbf{K}$ is called a *right circular cone*. The point $(0,0,0)$ is the vertex. The point $(.6, -.8, 1)$ is on the cone, and the entire line through that point and the vertex, $(0,0,0)$, is on the cone as well.

**Torus** A *torus* is a set that is a smooth version of the surface of a donut or a bagel. For example, the set $\mathbf{T}$ parametrized by

$$\gamma\begin{pmatrix} \theta \\ \phi \end{pmatrix} = \begin{pmatrix} (2+\cos\phi)\cos\theta \\ (2+\cos\phi)\sin\theta \\ \sin\phi \end{pmatrix} \quad \text{for} \quad 0 \le \theta < 2\pi \quad \text{and} \quad 0 \le \phi < 2\pi$$

is a torus.

**1.** Let $F$ be defined by

$$F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x - 3y + z + 3 \\ x + 2y - 2z - 3 \end{pmatrix}$$

and let $L$ be the curve (line) defined by $F = 0$ (which is the intersection of two planes).

(a) Use the implicit function theorem to show that $L$ is a smooth 1-manifold.

(b) Find a parametric representation for $L$.

**2.** Let $\gamma$ be defined by

$$\gamma(t) = \begin{pmatrix} 2t + 4 \\ -3t + 2 \\ t - 3 \end{pmatrix}$$

(a) Use the definition of "Parametrization of a manifold" (3.1.18 or 3.1.19) to show that $\gamma$ gives a parametrization of a smooth 1-manifold (line) $M$.

(b) Find a function $G$, as in **1.** above, that gives $M$ as the intersection of two planes. If you can, find $G$ so that one of the planes is parallel to the $z$-axis (i.e. does not intersect the $z$-axis) and the other is parallel to the $y$-axis.

**3.** Let $\mathbf{K}$ be the right circular cone defined above.

(a) Use the definition of "Parametrization of a manifold" to show that for each point of $\mathbf{K} \setminus \{0\}$, there is an open set $U$ in $\mathbb{R}^2$ so that for $(t, \theta)$ in this open set, $\gamma$ (or a slight modification of $\gamma$) gives a parametrization of a part of $\mathbf{K}$ containing that point so you have shown $\mathbf{K} \setminus \{0\}$ is a smooth 2-manifold.

(b) Find a function $f$ so that $f(x, y, z) = 0$ determines the set $\mathbf{K}$ and use the implicit function theorem to show that $\mathbf{K} \setminus \{0\}$ is a smooth 2-manifold. What happens at the origin?

**4.** Let $\mathbf{T}$ be the torus defined above.

(a) Use the definition of "Parametrization of a manifold" to show that for each point of $\mathbf{T}$, there is an open set $U$ in $\mathbb{R}^2$ so that for $(\theta, \phi)$ in this open set, $\gamma$ (or a slight modification of $\gamma$) gives a parametrization of a part of $\mathbf{T}$ containing that point so you have shown $\mathbf{T}$ is a smooth 2-manifold.

(b) Show that each plane containing the $z$-axis intersects the torus $\mathbf{T}$ in two circles. What are the radii of these circles? What is the distance from the centers of each of the circles to the $z$-axis?

(c) Show that the centers of these circles (which are not on $\mathbf{T}$ but inside of it) all lie on a circle that is a sort of circular 'axis' of the torus. Describe this circle, including the center, radius, and the plane containing it.

**5.** Let $\mathbf{S}$ be the set parametrized by the function $\gamma$ defined by

$$\gamma\begin{pmatrix} \theta \\ \phi \end{pmatrix} = \begin{pmatrix} 3\cos\theta\cos\phi + 2 \\ 3\sin\theta\cos\phi + 1 \\ 3\sin\phi - 2 \end{pmatrix} \quad \text{for } \ 0 \le \theta < 2\pi \ \text{ and } \ 0 \le \phi < \pi$$

(a) Use the definition of "Parametrization of a manifold" to show that for each point of $\mathbf{S}$, there is an open set $U$ in $\mathbb{R}^2$ so that for $(\theta, \phi)$ in this open set, $\gamma$ (or a slight modification of $\gamma$) gives a parametrization of a part of $\mathbf{S}$ containing that point so you have shown $\mathbf{S}$ is a smooth 2-manifold.

(b) Find a function $F$ that describes $\mathbf{S}$ as the set of points for which $F = 0$, and identify $\mathbf{S}$ as a sphere. What are the center and radius of $\mathbf{S}$?

**6.** The point $(-1.68, 2.24, .6)$ is on the torus $\mathbf{T}$ defined above; it corresponds to the angle $\theta$ in the second quadrant with $\sin\theta = .8$ and $\phi$ in the first quadrant with $\sin\phi = .6$. Find an equation for the plane tangent to the torus at that point.