Math 44500 (Cowen) **Topics & Terminology** **February 12, 2014**

Metric Spaces

- **Definition:** A *metric space* is a pair (X, d) where X is a non-empty set and $d : X \times X \mapsto [0, \infty)$ satisfies $d(x, x) = 0$ for all x in X, $d(x, y) = d(y, x) > 0$ for $x \neq y$ in X, and $d(x, z) \leq d(x, y) + d(y, z)$ for x, y, and z in X.
- **Definition:** In a metric space (X, d), $U \subset X$ *is an open set* if $U = \emptyset$ or for each x in U, there is $\epsilon > 0$ such that $\{y \in X : d(x, y) < \delta\} \subset U$. A set $F \subset X$ is called *closed* if $X \setminus F$ is an open set in X.
- **Definition:** We say a *sequence* (x_n) *in* X*, a metric space, converges to* y if for each $\epsilon > 0$, there is N so that $n > N$ implies $d(x_n, y) < \epsilon$.
- **Theorem:** If F is a subset of X, a metric space, then F is closed if and only if for every convergent sequence (x_n) of points in F, say $\lim_{n\to\infty} x_n = y$, then y is in F, also.
- **Definition:** We say *the sequence* (x_n) *in* X*, a metric space, is a Cauchy sequence* if for each $\epsilon > 0$, there is N so that $m, n > N$ implies $d(x_n, x_m) < \epsilon$.
- **Definition:** We say *the metric space* X *is complete* if every Cauchy sequence in X converges.
- **Examples:** $\mathbb{R}$ and $\mathbb{C}$ are complete metric spaces with $d(w, z) = |w - z|$ and $\mathbb{R}^n$ and $\mathbb{C}^n$ are complete metric spaces with $d(w, z) = \|w - z\|$.
- **Definition:** We say G*, a subset of the metric space* X*, is connected* if there are no open subsets U and V of X so that $U \cap V = \emptyset$ but $G \cap U$ and $G \cap V$ are both non-empty.
- **Theorem:** In $\mathbb{R}$, a set is connected if and only if it is a finite or infinite interval.
- **Definition:** We say K*, a subset of the metric space* X*, is compact* if for every collection of open sets $\{U_j\}_{j\in J}$ with $K \subset \bigcup_{j\in J} U_j$ (an *open cover of* K, there are finitely many $U_{j_1}, U_{j_2}, \cdots, U_{j_n}$ such that $K \subset \bigcup_{k=1}^{n} U_{j_k}$ (a *finite subcover*).
- **Theorem:** If X is a metric space and K is a compact subset of X, then K is a closed and bounded set. If K is a compact subset of X and F is a subset of K that is closed in X, then F is compact.
- **Theorem:** In $\mathbb{R}^n$ or $\mathbb{C}^n$, a subset K is a compact subset if and only if K is a closed and bounded set.
- **Definition:** If X and Y are metric spaces and $f : X \mapsto Y$ is a function mapping X into Y, we say f *is continuous* if for every open subset U in Y, the set $f^{-1}(U)$ is open in X.
- **Theorem:** If f is a function mapping the metric space X into Y, then f is continuous if and only if for every closed subset F of Y, the set $f^{-1}(F)$ is closed in X.
- **Theorem:** If f is a function mapping the metric space X into Y, then f is continuous if and only if for (x_n) a sequence in X with $\lim_{n\to\infty} x_n = z$ then $\lim_{n\to\infty} f(x_n) = f(z)$ in Y.
- **Theorem:** If f is a continuous function mapping the metric space X into Y and K is a compact subset of X, then $f(K)$ is a compact subset of Y.
- **Theorem:** If f is a continuous function mapping the metric space X into Y and A is a connected subset of X, then $f(A)$ is a connected subset of Y.
- **Corollary:** If f is a continuous, real valued function on the interval $[a, b] \subset \mathbb{R}$, then there are numbers c_1 and c_2 in $[a, b]$ so that $f([a, b]) = [f(c_1), f(c_2)]$.

- **Definition:** If X and Y are metric spaces and $f : X \mapsto Y$ is a function mapping X into Y, we say f *is injective or 1-to-1* if, for x_1 and x_2 points of X, $f(x_1) = f(x_2)$ implies $x_1 = x_2$ and we say f *is surjective or onto* if for each y in Y, there is a point x of X for which $f(x) = y$. If f is injective and surjective, defining $g : Y \mapsto X$ by $g(y) = x$ if $f(x) = y$, then $g(f(x)) = x$ for all x in X and $f(g(y)) = y$ for all y in Y. In this case, the function g is called the *inverse function for* f and we write $g = f^{-1}$. This introduces possible confusion between f^{-1} as a set function and f^{-1} as a point function, but the relation between them, if they both exist, is $f^{-1}(\{y\}) = \{g(y)\} = \{f^{-1}(y)\}$.
- **Theorem:** Suppose X is a compact metric space and Y is a metric space. If f is a continuous, injective and surjective function mapping X onto Y, then Y is compact and f^{-1} is a continuous function mapping Y onto X.

Null **Subsets or Subsets** ***of Measure Zero*** **of** $\mathbb{R}$

- A subset F of $\mathbb{R}$ is called a *null set* or *set of measure zero* if for each $\epsilon > 0$, there are real numbers $a_n < b_n$ for $n = 1, 2, 3, \cdots$ such that $\sum_n |b_n - a_n|$ and $F \subset \bigcup_{n=1}^{\infty} (a_n, b_n)$, that is F can be covered by a open intervals of arbitrarily small *total* length.
- **Examples:** For r a real number, $\{r\}$ is a set of measure zero and $[0, 1]$ is set that is *not* measure zero.
- **Theorem:** A countable union of sets of measure zero is a set of measure zero.
- **Corollary:** Every countable subset of the real numbers is a set of measure zero.
- **Examples:** The set $\mathbb{Q}$ of rational numbers is a set of measure zero. The set

  $$\Omega = \{x \in \mathbb{R} : \ x \text{ has a decimal expansion consisting of only 0's and 1's}\}$$

  is an uncountable set of measure zero and $[0, 1]$ is an uncountable set that is *not* measure zero.