Math 44500 (Cowen) **Handout 9** **18 April 2011**

Due Thursday, 21 April:

Definition: Suppose (f_n) is a sequence of functions. We say the sequence (f_n) is *equicontinuous* if, for each $\epsilon > 0$, there is $\delta > 0$ such that if $|u-v| < \delta$ then $|f_n(u)-f_n(v)| < \epsilon$ for every n.

A. Prove that if (f_n) is an equicontinuous sequence of functions such that $\lim_{n\to\infty} f_n(x) = 0$ for every x in $[0,1]$ then the sequence (f_n) converges uniformly to 0 on $[0,1]$.

Definition: For x in $\mathbb{R}$, let

$$\cos(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} \quad \text{and} \quad \sin(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

(Easy calculations (that you do not need to duplicate) show that these series converge absolutely for all x in $\mathbb{R}$ and uniformly on every compact subset of $\mathbb{R}$, so the definitions make sense.)

B. Using the definitions above, prove that, for every x in $\mathbb{R}$,

$$\frac{d}{dx}\cos(x) = -\sin(x) \quad \text{and} \quad \frac{d}{dx}\sin(x) = \cos(x)$$

From these equalities, conclude that the function

$$k(x) = (\sin(x))^2 + (\cos(x))^2$$

is actually a constant and, using the series above, find the constant.

C. By multiplying the series for $\sin(x)$ by the series for $\cos(x)$ show that $\sin(2x)) = 2\sin(x)\cos(x)$ for all x in $\mathbb{R}$.

Hint: If k is a positive integer,

$$\begin{aligned}(a+b)^{2k+1} &= \sum_{j=0}^{2k+1} \frac{(2k+1)!}{j!(2k+1-j)!} a^{2k+1-j} b^j \\ &= \sum_{\ell=0}^{k} \frac{(2k+1)!}{(2\ell)!(2(k-\ell)+1)!} a^{2(k-\ell)+1} b^{2\ell} + \frac{(2k+1)!}{(2\ell+1)!(2(k-\ell))!} a^{2(k-\ell)} b^{2\ell+1}\end{aligned}$$

so for $a = b = 1$, we get

$$\begin{aligned}2^{2k+1} &= \sum_{\ell=0}^{k} \frac{(2k+1)!}{(2\ell)!(2(k-\ell)+1)!} + \frac{(2k+1)!}{(2\ell+1)!(2(k-\ell))!} \\ &= 2\sum_{\ell=0}^{k} \frac{(2k+1)!}{(2\ell+1)!(2(k-\ell))!}\end{aligned}$$

which means

$$\frac{2^{2k+1}}{(2k+1)!} = 2\sum_{\ell=0}^{k} \frac{1}{(2\ell+1)!(2(k-\ell))!}$$