FOR DISCUSSION Thursday, 24 February:

From page 155 of **BS**: 5, 6, 10, 12

From page 182 of **BS**: 3, 4, 5, 7cd, 9ab

A. For $x > 0$, define the function L by

$$L(x) = \int_1^x \frac{1}{t}\,dt$$

(For $0 < x < 1$, the integral $\int_1^x \frac{1}{t}\,dt$ is interpreted as $-\int_x^1 \frac{1}{t}\,dt$, as usual.)

(a) Prove, using the Fundamental Theorem of Calculus, that the function L is continuous and differentiable on $(0,\infty)$. Find $L(1)$. Use an easy Riemann sum estimate to show that $L(2) < 1$ and $L(4) > 1$.

(b) Prove that L is strictly increasing on $(0,\infty)$. It is a fact that $\lim_{x\to\infty} L(x) = \infty$ and that $\lim_{x\to 0^+} L(x) = -\infty$, so that L is a continuous and differentiable function that maps $(0,\infty)$ onto $\mathbb{R}$.

(c) Prove that if $a > 1$ and $b > 1$, then $L(ab) = L(a) + L(b)$. (**Hint:** Use

$$L(ab) = \int_1^{ab} \frac{1}{t}\,dt = \int_1^a \frac{1}{t}\,dt + \int_a^{ab} \frac{1}{t}\,dt$$

and use the substitution $u = t/a$ to change variables in the second integral.)

(d) Since L is a strictly increasing differentiable function mapping $(0,\infty)$ onto $\mathbb{R}$, it has a continuous and differentiable inverse function: let E be the inverse function for L so that E is a differentiable function that maps $\mathbb{R}$ onto $(0,\infty)$ and satisfies $E(L(x)) = x$ and $L(E(y)) = y$. Find $E(0)$. Show that $2 < E(1) < 4$.

(e) Use the theorems we know to find the derivative of E, that is, find $E'(y)$ for y in $\mathbb{R}$.

(f) Use part (c) above to show that $E(p+q) = E(p)E(q)$.