Math 44500 (Cowen) **Handout 2** **13 January 2011**

Due Thursday, 20 January:

A. Suppose A is a subset of $\mathbb{R}^n$. Show that a is a boundary point of A if and only if there is a sequence (a_n) consisting of points in A such that $\lim_{n\to\infty} a_n = a$ AND a sequence (b_n) consisting of points in the complement of A such that $\lim_{n\to\infty} b_n = a$.
(Note: constant sequences are OK.)

B. Suppose G is a subset of $\mathbb{R}^n$. Show that G is an open set if and only if it does not contain any of its boundary points.

C. Suppose F is a subset of $\mathbb{R}^n$. Show that F is a closed set if and only if it contains all of its boundary points.

D. For x and y real numbers, let $d(x,y) = |\arctan(x) - \arctan(y)|$.

(a) Show that d is a metric on $\mathbb{R}$, that is, $\mathbb{R}$ with the metric d is a metric space.
(Hint: you may want to prove that arctan is an increasing function on $\mathbb{R}$ and use that fact to show that d is a metric.)

(b) If we say a set K is bounded if $\sup\{d(x,y) : x, y \in K\}$ is finite, show that $\mathbb{R}$ is a bounded set with this metric.

(c) Give an example of a set, in the metric space $(\mathbb{R}, d)$ that is closed and bounded, (in this sense) but not compact. This shows that the characterization of compact sets in $\mathbb{R}^n$ as the closed and bounded sets does not extend to all metric spaces.

E. Suppose $K \subset \mathbb{N}$, regarded as a subset of $\mathbb{R}$. Show that K is compact if and only if K is finite.

NOTE: correction in the definition of connected:
A set S is connected there are NOT open sets U and V so that $U \cap V = \emptyset$, $U \cap S \neq \emptyset$, $V \cap S \neq \emptyset$, and $S \subset U \cup V$

F. Let $S = \{(x,y) \in \mathbb{R}^2 : x \leq 0 \quad \text{OR both} \quad y > 0 \ \text{ and } xy \geq 1\}$

(a) Prove that S is a closed set in $\mathbb{R}^2$.

(b) Show that S is not connected, that is, find open sets U and V in $\mathbb{R}^2$ so that $S \subset U \cup V$ and $U \cap V = \emptyset$.