Math 44500 (Cowen) **Handout 10** **21 April 2011**

Introduction to Measure Theory

GOAL: Assign a numerical 'size' to every set: a function that does something like this will be called a *measure*, μ. We want a measure to satisfy:

- The measure of S, $\mu(S)$, satisfies $0 \leq \mu(S) \leq \infty$ for each set S. (If μ is a measure defined for subsets of a set X, and $\mu(X) < \infty$, we call μ a *finite measure*.)
- $\mu(\emptyset) = 0$.
- If $S \subset T$, then $\mu(S) \leq \mu(T)$.
- If S_1, S_2, S_3, $\cdots$ are disjoint sets, then (*countable additivity*)

$$\mu(\bigcup_{j=1}^{\infty} S_j) = \sum_{j=1}^{\infty} \mu(S_j)$$

The BAD NEWS: For most interesting situations, this goal cannot be achieved!

So we sacrifice the 'every set' in the statement of the goal: in most interesting situations, we can achieve the goal for a large collection of sets. If μ is a measure on the set X, a subset S is called *measurable* (or *μ-measurable*) if $\mu(S)$ is defined. At the very least, we have $\emptyset$ and X are measurable and we want S and T measurable to imply $S \cup T$ and $S \cap T$ are measurable and, if $\mu(X) < \infty$, then $X \setminus S$ is also measurable.

New GOAL: Define *Lebesgue measure*, μ, on $\mathbb{R}$. The collection of Lebesgue measurable sets includes the open subsets of $\mathbb{R}$, the compact subsets of $\mathbb{R}$, and countable unions of these sets.

Outline:

- **Definition** If a and b are real numbers with $a < b$, then $\mu((a, b)) = b - a$ and if $a = -\infty$ or $b = \infty$ or both, then $\mu((a, b)) = \infty$.
- Every non-empty open subset of $\mathbb{R}$ is a countable union of disjoint open intervals.
- **Definition** If U is an open set and $U = \bigcup_{j=1}^{\infty} I_j$ where the I_j are disjoint open intervals, then $\mu(U) = \sum_{j=1}^{\infty} \mu(I_j)$.
- If U and V are open sets with $U \subset V$ and $U \neq V$, then either $\mu(U) = \mu(V) = \infty$ or $\mu(U) < \mu(V)$.
- Suppose I_j with $j = 1, 2, 3 \cdots$ is a sequence of open intervals with $I_j \subset I_{j+1}$ for every j in $\mathbb{N}$. Then $\mu(\bigcup_{j=1}^{\infty} I_j) = \lim_{j\to\infty} \mu(I_j)$.
- Suppose U_j with $j = 1, 2, 3 \cdots$ is a sequence of open sets with $U_j \subset U_{j+1}$ for every j in $\mathbb{N}$. Then $\mu(\bigcup_{j=1}^{\infty} U_j) = \lim_{j\to\infty} \mu(U_j)$.
- **Definition** If S is a subset of the real numbers, then the *outer measure of* S, denoted $\overline{\mu}(S)$, is

$$\overline{\mu}(S) = \inf\{\mu(U) : U \supset S \text{ and } U \text{ is open}\}$$

- If S and T are subsets of the real numbers and $S \subset T$, then $\overline{\mu}(S) \leq \overline{\mu}(T)$.
- Suppose S_j with $j = 1, 2, 3 \cdots$ is a sequence of subsets with $S_j \subset S_{j+1}$ for every j in $\mathbb{N}$. Then $\overline{\mu}(\bigcup_{j=1}^{\infty} S_j) = \lim_{j\to\infty} \overline{\mu}(S_j)$.
- Suppose K is a non-empty compact set and $a = \inf\{x : x \in K\}$ and let $b = \sup\{x : x \in K\}$. If $U = [a, b] \setminus K$, then

$$\overline{\mu}(K) = b - a - \mu(U)$$

Outline(cont'd):

- **Definition** If K is a compact subset of the real numbers, then the *measure of* K is defined to be $\mu(K) = \overline{\mu}(K)$.
- If K and L are compact subsets of the real numbers and $K \subset L$, then $\mu(K) \leq \mu(L)$.
- **Definition** If S is a subset of the real numbers with $\overline{\mu}(S) = 0$, then we say S *is a set of measure zero* or S *is a null set* and we write $\mu(S) = 0$.
- **Definition** If $\mathcal{Q}(x)$ is a statement about the point x in I, then we say $\mathcal{Q}(x)$ *is true almost everywhere on* I or $\mathcal{Q}(x)$ *is true for almost every* x *in* I if there is set $X \subset I$ with $\mu(X) = 0$ and $\mathcal{Q}(x)$ is true for all x in $I \setminus X$.
- If S is a countable set, then S is a set of measure zero.
- Almost every real number is irrational.
- If f is an increasing function on the interval I, then f is continuous almost everywhere on I.
- Let C be the Cantor middle thirds set. Then C is a compact, uncountable set of measure zero.
- Let F be the Cantor 'middle fourths set', constructed in the same way as the middle thirds set but removing the segment $[3/8, 5/8]$ in the first step, then two intervals of length $1/16$, etc. Then there is a continuous, strictly increasing function of the interval $[0, 1]$ onto itself such that $f(C) = F$ but $\mu(F) = 1/2$. That is, homeomorphisms are *not* measure preserving!
- **Definition** If S is a subset of the real numbers, then the *inner measure of* S, denoted $\underline{\mu}(S)$, is

$$\underline{\mu}(S) = \sup\{\mu(K) : K \subset S \text{ and } K \text{ is compact}\}$$

- If U is an open subset of $\mathbb{R}$, then $\underline{\mu}(U) = \mu(U)$.
- If S and T are subsets of $\mathbb{R}$ with $S \subset T$, then $\underline{\mu}(S) \leq \underline{\mu}(T)$.
- If S is any subset of $\mathbb{R}$, then $\underline{\mu}(S) \leq \overline{\mu}(S)$.
- **Definition** If S is a subset of the real numbers with $\overline{\mu}(S) < \infty$, then we say S *is measurable* or S *is Lebesgue measurable* if $\underline{\mu}(S) = \overline{\mu}(S)$ and in this case, we say the *measure (or the Lebesgue measure) of* S *is* $\underline{\mu}(S) = \overline{\mu}(S)$. If $\overline{\mu}(S) = \infty$, we will say S *is measurable* if each of the sets $[-n, n] \cap S$ is measurable for n a positive integer.
- If S and T are measurable subsets of $\mathbb{R}$ with $S \subset T$, then $\mu(S) \leq \mu(T)$.
- For a and b real numbers with $a \leq b$, the intervals $[a, b]$, $[a, b)$, $(a, b]$, and (a, b) are Lebesgue measurable and all have measure $b - a$.
- If S_1, S_2, S_3, $\cdots$ are disjoint, measurable sets, then (*countable additivity*)

$$\mu\left(\bigcup_{j=1}^{\infty} S_j\right) = \sum_{j=1}^{\infty} \mu(S_j)$$

- Using the Axiom of Choice, it is possible to show there are subsets of $\mathbb{R}$ that are not Lebesgue measurable. (See, for example, *Measure Theory* by P. R. Halmos, 1950, pages 69-70.)