Math 444 (Cowen) **Homework** **Due 1 October 2014**

3rd: page 74: 1, 3, 4, 11 **and** **3rd: page 80:** 1, 5, 11, 12
4th: page 77: 1, 3, 4, 10 **and** **4th: page 84:** 1, 5, 11, 12

A. Do problem 9 in the problem set for Section 3.3 of the text. Then answer the following:

(a) Let B be the set $B = \{3 + 1/k : k \in \mathbb{N}\}$. What is $\sup B$? If $A = B$ what does your proof of the result in problem 9 give for (x_n), or if your proof is not constructive, find an sequence that satisfies the conditions of problem 9.

(b) Let C be the set

$$C = \{7 - 3/j - 4/k : j \text{ is an even number in } \mathbb{N}, \text{ and } k \text{ is an odd number in } \mathbb{N}\}$$

What is $\sup C$? If $A = C$ what does your proof of the result in problem 9 give for (x_n), or if your proof is not constructive, find an sequence that satisfies the conditions of problem 9.

B. Define a sequence recursively by $c_1 = 5$ and $c_{n+1} = 3 - 1/c_n$.

(a) Prove (by induction) that $c_n \geq 1$ for all n in $\mathbb{N}$.

Lemma. *For each n in $\mathbb{N}$ with $n \geq 2$, we have $c_n - c_{n-1} \leq 0$.*

Proof: The number c_1 is 5 and the number c_2 is 2.8 which is less than $c_1 = 5$, and $c_2 - c_1 \leq 0$. We will use induction to show that for each n in $\mathbb{N}$ with $n \geq 2$, then $c_n - c_{n-1} \leq 0$. The inequality above, $c_2 - c_1 \leq 0$, is the case $n = 2$.
Suppose the result is true for $n = k$, that is, that $c_k - c_{k-1} \leq 0$, we will show that $c_{k+1} - c_k \leq 0$. We have $c_{k+1} - c_k = 3 - 1/c_k - c_k$. On the other hand, $c_k = 3 - 1/c_{k-1}$, so the expression above is

$$c_{k+1} - c_k = 3 - \frac{1}{3 - 1/c_{k-1}} - (3 - 1/c_{k-1}) = 3 - \frac{1}{3 - 1/c_{k-1}} - 3 + \frac{1}{c_{k-1}}$$

$$= \frac{1}{c_{k-1}} - \frac{1}{3 - 1/c_{k-1}} = \frac{(3 - 1/c_{k-1}) - c_{k-1}}{c_{k-1}(3 - 1/c_{k-1})} = \frac{c_k - c_{k-1}}{(3c_{k-1} - 1)}$$

Since $c_{k-1} \geq 1$ by part (a), it follows that $(3c_{k-1} - 1) \geq 0$. This means that $c_k - c_{k-1} \leq 0$ implies $c_{k+1} - c_k \leq 0$ and the result is true for $n = k + 1$ also. Thus, the Lemma is true by induction.

(b) Prove that $\lim_{n\to\infty} c_n$ exists and find the limit.