Professor Carl Cowen **Math 42500-52500** **Fall 18**

PROBLEMS

September 12:

18. Let $f(z) = z^4$. Write z as $z = x + iy \sim (x, y)$ and write f as

$$f(z) = u(z) + iv(z) = u(x + iy) + iv(x + iy) \sim u(x, y) + iv(x, y)$$

(a) Find $u(x, y)$ and $v(x, y)$ as functions of x and y.
(b) Use the Cauchy-Riemann equations (and the fact that the relevant functions are continuous) to show that $f(z) = z^4$ is analytic.

19. For which real numbers a, b, c, and d is the function $u(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$ harmonic on $\mathbb{C}$? For the cases in which u is harmonic, find a harmonic conjugate, v, of u.

20. Let $u(x, y) = \dfrac{x^2 + y^2 - 1}{(x - 1)^2 + y^2}$
(a) Show that $u(x, y)$ is a harmonic function on $\mathbb{R}^2 \setminus \{(1, 0)\}$
(b) Find a harmonic conjugate v for u on the same domain.
(c) Find a function $f(z)$ that is analytic on $\mathbb{C} \setminus \{1\}$ such that u is the real part of f and v is the imaginary part of f.

21. Let $a \neq 0$, b, and c be complex numbers, and let $p(z) = az^2 + bz + c$. The quadratic formula from high school was *The roots of p are the numbers:*

$$r_+ = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad r_- = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Show that the quadratic formula works for complex numbers also by showing that p can be factored as

$$p(z) = a(z - r_+)(z - r_-)$$

22. Let $F(z) = \dfrac{z^3 + 2z + 5}{z^4 - 3z^2 - 4}$. (Note that the denominator of F can be factored as $(z^2 - 4)(z^2 + 1)$.) Find the partial fractions decomposition of F; that is, find complex numbers a, b, c, d, p, q, r and s so that:

$$F(z) = \frac{a}{z - p} + \frac{b}{z - q} + \frac{c}{z - r} + \frac{d}{z - s}$$