Homework 7

Note: Each of the following questions has appeared on Test I in a previous Math 35100 class.

1. $3x + 2y - z = 0$ is the equation of a plane $\mathcal{P}$ in $\mathbb{R}^3$ that passes through the origin, so $\mathcal{P}$ is a subspace of $\mathbb{R}^3$. Find a basis for $\mathcal{P}$. What is the dimension of this subspace?

2. Let $F = \begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & -3 \\ -1 & 8 & -9 \end{pmatrix}$
   (a) Find a basis for the column space of F.
   (b) Find a basis for the column space of F'.
   (c) Is $(1, 1, 1)$ in the column space of F?

3. Let $A = \begin{pmatrix} 1 & -1 & 1 & 0 & -2 \\ 2 & 0 & 1 & 1 & -1 \\ 1 & -3 & 2 & -1 & -5 \end{pmatrix}$
   (a) Find a basis for the column space of A.
   (b) Find a basis for the null space of A.
   (c) Is $(1, 1, 1)$ in the column space of A?

4. Let $B = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & -3 \\ 3 & 1 & 5 \\ 2 & -1 & 5 \end{pmatrix}$
   (a) Find a basis for the column space of B.
   (b) Find a basis for the column space of B'.
   (c) Is $(1, 3, -1, -4)$ in the column space of B?

5. (a) Show that the vectors $(1, 2, 1)$, $(-1, 1, 0)$, and $(2, 0, 0)$ are a basis for $\mathbb{R}^3$.
   (b) Find the coordinates of $(0, 0, 1)$ with respect to this basis.

6. (a) The vectors $u = (2, 1, -1)$, $v = (1, 1, 0)$, and $w = (1, 2, 1)$ are linearly dependent in $\mathbb{R}^3$.
   (b) Write $z = u + v - 2w = (1, -2, -3)$ as a linear combination of two or fewer of these vectors.

7. Find a basis for the subspace of $\mathbb{R}^4$ spanned by $(1, -1, 5, -5)$, $(1, 1, -1, 1)$, and $(2, 1, 1, -1)$. What is the dimension of this subspace?

8. Find a basis for the subspace of $\mathbb{R}^4$ spanned by $(1, -1, 1, 2)$, $(1, 1, -1, 1)$, and $(2, 1, 1, -1)$. What is the dimension of this subspace?

For each of the following matrices, find bases for the range and the nullspace, and find the dimensions of these subspaces.

9. $\begin{pmatrix} 1 & -1 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & -1 & 4 & 0 \end{pmatrix}$

10. $\begin{pmatrix} 1 & -1 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & -1 & 1 \\ 3 & 3 & -1 & 2 \end{pmatrix}$

11. For each of the situations (a)–(f) below, decide which of the statements in the box can correctly complete the sentence. *Include all correct responses.*
   (a) If A is an 8×11 matrix whose rank is 6, then __________
   (b) If A is an 8×11 matrix whose rank is 8, then __________
   (c) If A is an 8×11 matrix whose rank is 10, then __________
   (d) If A is a 12×7 matrix whose rank is 9, then __________
   (e) If A is a 12×7 matrix whose rank is 7, then __________
   (f) If A is a 12×7 matrix whose rank is 5, then __________

(i) $AX = b$ is solvable for every vector b.
(ii) there are some vectors b for which $AX = b$ is not solvable.
(iii) for some vectors b, the system $AX = b$ has exactly one solution.
(iv) for some vectors b, the system $AX = b$ has infinitely many solutions.
(v) the given information is contradictory, no such system is possible.

12.
$$C = \begin{pmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 1 & 2 \\ 2 & -1 & 5 & -4 \end{pmatrix}$$
The vectors $v_1 = (-3, -1, 1, 0)$ and $v_2 = (1, -2, 0, 1)$ are a basis for $\mathcal{N}(C)$, the nullspace of C.
   (a) What is the dimension of $\mathcal{N}(C)$, the null space of C?
   What is the rank of C?
   What is the rank of C'?
   What is the dimension of $\mathcal{N}(C')$?
   (b) Find bases for the range of C, the range of C', and the nullspace of C'.
   (c) Is the equation $CX = (2, 3, 1)$ solvable?
   (d) Is the equation $CX = (-1, 1, 3)$ solvable?
   (e) Is the equation $C'X = (1, -1, 2, -3)$ solvable?

13.
$$D = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 1 & 3 & -1 \end{pmatrix}$$
   (a) Find a basis for the nullspace of D.
   (b) What is the nullity of D? the rank of D? the rank of D'? the nullity of D'?
   (c) Find bases for the range of D, the range of D', and the nullspace of D'.
   (d) Is the equation $DX = (2, 3, 1)$ solvable?
   (e) Is the equation $DX = (1, 1, 2)$ solvable?
   (f) Is the equation $D'X = (1, 0, 0, -1)$ solvable?
   (g) Is the equation $D'X = (1, 2, 1, -1)$ solvable?

14. For each real number t, let $F(t)$ be the matrix
$$F(t) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ t & 2 & 1 & 1 \\ 1 & -2 & t & -1 \\ 0 & 3 & 1 & 2 \end{pmatrix}$$
   (a) Find $\det(F(t))$. (It is a function of t.)
   (b) For which value (or values) of t are the columns of $F(t)$ linearly dependent?

15. For each of the following, decide if the statement is *always true* or *always false* or *sometimes true, sometimes false* when the given condition is true.

   (a) **Given:** The vectors v_1, v_2, v_3, v_4, v_5, v_6 span $\mathbb{R}^6$.
   Statement: The vectors v_1, v_2, v_3, v_4, v_5, v_6 are linearly independent.

   always true   always false   sometimes true, sometimes false

   (b) **Given:** B is a 6×6 matrix with $\det(B) \neq 0$.
   Statement: The equation $BX = b$ has infinitely many solutions.

   always true   always false   sometimes true, sometimes false

   (c) **Given:** B is a 6×6 matrix, b is in $\mathbb{R}^6$, $BX = 0$ has infinitely many solutions.
   Statement: The equation $BX = b$ has infinitely many solutions.

   always true   always false   sometimes true, sometimes false

   (d) **Given:** B is a 6×6 matrix and the columns of B are linearly independent.
   Statement: The equation $BX = b$ has exactly one solution.

   always true   always false   sometimes true, sometimes false

   (e) **Given:** E is a 6×8 matrix, b is in $\mathbb{R}^6$, $\mathcal{N}(E)$, the nullspace of E is 2–dimensional.
   Statement: The equation $EX = b$ has infinitely many solutions.

   always true   always false   sometimes true, sometimes false

   (f) **Given:** D is a 7×5 matrix, b is in $\mathbb{R}^7$, $\mathcal{N}(D)$, the nullspace of D is 1–dimensional.
   Statement: The equation $DX = b$ has infinitely many solutions.

   always true   always false   sometimes true, sometimes false

16. Consider the system:

$$\left\{\begin{array}{rcr} u + 2v + w - x - 2y & = & 3 \\ -2u + v + w + x + 2y & = & 5 \\ u + v - w + 2x + 4y & = & -2 \\ u - v \qquad + 3x + y & = & -7 \\ -u + 3v + w + x + 3y & = & 7 \end{array}\right.$$

   (a) Choose A and b so that the system can be written in matrix form as $AX = b$ where $X = (u, v, w, x, y)$.
   (b) Check that $X_p = (-1, 1, 2, -2, 1)$ is a solution of the system and check that $X_0 = (-1, 1, -2, 1, -1)$ is a solution of the associated homogeneous system $AX = 0$.
   (c) Without using Gaussian elimination or a machine, find two other non-trivial solutions of $AX = 0$.
   (d) Without using Gaussian elimination or a machine, find two other solutions of $AX = b$.

17. The five-tuples $(2, 2, 1, -1, 1)$ and $(1, 1, 2, -1, -1)$ are both solutions of the system:

$$\left\{\begin{array}{rcl} a + b + 4c + d + e & = & 8 \\ a - b + 2c + 2d + e & = & 1 \\ 2a + b - c - d - 2e & = & 4 \\ b + 3c + d + e & = & 5 \\ 2a - b + c + 3d \qquad & = & 0 \end{array}\right.$$

   (a) Without using Gaussian elimination or a machine, write down two non-trivial solutions of the associated homogeneous system.
   (b) Write down two other solutions of the given system.