Homework 6

1. In answering a question on her linear algebra homework, April claimed that the subspace $\mathcal{W}$ is spanned by the set $u_1 = (1, 0, 1)$ and $u_2 = (0, 1, -1)$. Michelle claimed that the subspace $\mathcal{W}$ is spanned by $v_1 = (1, 1, 0)$, $v_2 = (2, 1, 1)$, and $v_3 = (1, -1, 2)$. Do their answers agree with each other, that is, is the subspace spanned by the set $\{u_1, u_2\}$ the same as the subspace spanned by $\{v_1, v_2, v_3\}$?

2. In answering a question on his linear algebra homework, Max claimed that the subspace $\mathcal{U}$ is spanned by the set $u_1 = (1, 0, 1, 1)$, $u_2 = (0, 1, -1, 0)$ and $u_3 = (0, 0, 1, 2)$. Spike claimed that the subspace $\mathcal{U}$ is spanned by $v_1 = (1, 1, 0, 1)$, $v_2 = (2, 1, 1, 2)$, and $v_3 = (1, -1, 2, 1)$. Do they agree with each other, that is, is the subspace spanned by the set $\{u_1, u_2, u_3\}$ the same as the subspace spanned by $\{v_1, v_2, v_3\}$?

3. Let $B = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 2 \\ -1 & 0 & 1 \end{pmatrix}$.
   (a) Taking $X = (1, 1, -2)$ shows that $(4, -3, -3) = BX$ is in the range of B. Show that $(4, -3, -3)$ is a linear combination of $(1, 2, -1)$, $(1, -1, 0)$, and $(-1, 2, 1)$.
   (b) The vector $v = (1, 2, -1) + 2(1, -1, 0) - (-1, 2, 1) = (4, -2, -2)$ is a linear combination of the columns of B. Find X so that $BX = v$.
   (c) Is the vector $(-2, 11, 1)$ in the subspace spanned by $(1, 2, -1)$, $(1, -1, 0)$, and $(-1, 2, 1)$?

4. In doing his linear algebra homework, Eduardo found that if $u = (2, 1, -3)$, $v = (1, -1, 2)$, and $w = (1, 5, -12)$, then $2u - 3v - w = 0$. Does this calculation show that u, v, and w form a linearly dependent set, that they form a linearly independent set, or does it not show either of these?

5. In doing her linear algebra homework, Pauline was given that $p = (1, 1, -3)$, $q = (2, -1, 2)$, and $r = (1, -2, 5)$. She noticed that $0p + 0q + 0r = 0$. Does this observation show that p, q, and r form a linearly dependent set, that they form a linearly independent set, or does it not show either of these?

   Decide if the following sets of vectors are linearly dependent or independent. If independent, prove that they are, if dependent, find a non-trivial linear combination of the vectors that gives zero.

6. $\{(2, 3, 1),\ (-1, 1, 2),\ (0, 1, 1)\}$

7. $\{(-1, 2, 1),\ (1, 1, 3),\ (1, 0, 1)\}$

8. $\{(0, 1, 1, -1),\ (1, 3, 1, -2),\ (2, 1, 0, -3),\ (3, 1, -1, 2),\ (2, -1, 2, 0)\}$

9. $\{(1, 1, -1, 2),\ (3, -1, 1, 1),\ (2, 0, -1, 1),\ (0, 2, -3, 2)\}$

10. In doing his linear algebra homework, Eduardo found that when $u = (2, 1, -3)$, $v = (1, -1, 2)$, and $w = (1, 5, -12)$, then $2u - 3v - w = 0$. Write one of these vectors as a linear combination of the other two.

11. The vectors $v_1 = (1, -1, 2)$, $v_2 = (-1, 2, -3)$, $v_3 = (1, 1, -1)$, and $v_4 = (-2, 3, -4)$ are linearly dependent in $\mathbb{R}^3$. Write one of the vectors as a linear combination of the rest.

12. The vector $u = (8, -7, 11)$ is a linear combination of the vectors $v_1 = (1, -1, 2)$, $v_2 = (-1, 2, -3)$, $v_3 = (1, 1, -1)$, and $v_4 = (-2, 3, -4)$ of Exercise 11, namely, $u = v_1 - v_2 + 2v_3 - 2v_4$. Use your answer to that exercise to write u as a linear combination of only three of the v's.

For each of the following sets of vectors, decide if they span $\mathbb{R}^3$. If they do, show that they do. If they do not, find a vector that is not in the subspace spanned by the vectors.

13. $\{(1,-1,0),\ (3,1,0)\}$
14. $\{(1,0,0),\ (-2,1,0),\ (1,1,-1)\}$
15. $\{(2,1,-1),\ (1,0,-1),\ (1,1,0),\ (0,1,1)\}$

16. Let A be an $m \times n$ matrix and let v_1, v_2, $\ldots$, v_k be vectors in $\mathbb{R}^n$. Show that if Av_1, Av_2, $\ldots$, Av_k are linearly independent vectors in $\mathbb{R}^m$, then v_1, v_2, $\ldots$, v_k are linearly independent in $\mathbb{R}^n$.

17. (a) Let A be an invertible $m \times m$ matrix and let v_1, v_2, $\ldots$, v_k be vectors in $\mathbb{R}^m$. Show that if v_1, v_2, $\ldots$, v_k are linearly independent vectors in $\mathbb{R}^m$, then Av_1, Av_2, $\ldots$, Av_k are also linearly independent.
    (b) Find an example of a non-invertible $m \times m$ matrix B and linearly independent vectors w_1, w_2, $\ldots$, w_k so that Bw_1, Bw_2, $\ldots$, Bw_k are linearly dependent.

18. Find a basis for the solution space of the system:

$$\begin{cases} a + 3b - c + 2d &= 0 \\ 2a + 2b + c + 2d &= 0 \\ 4a \qquad + 5c - 7d &= 0 \end{cases}$$

What is the dimension of this subspace?

19. Find a basis for the solution space of the system:

$$\begin{cases} u + 3v \ - w + 2x + y &= 0 \\ u + 2v + 4w + 2x &= 0 \\ 2u + 8v \ + w + 3x - y &= 0 \end{cases}$$

What is the dimension of this subspace?