Homework 7

For each of the following sets of vectors, decide if they span $\mathbb{R}^3$. If they do, show that they do. If they do not, find a vector that is not in the subspace spanned by the vectors.

1. $\{(1,-1,0),\ (3,1,0)\}$
2. $\{(1,0,0),\ (-2,1,0),\ (1,1,-1)\}$
3. $\{(2,1,-1),\ (1,0,-1),\ (1,1,0),\ (0,1,1)\}$
4. Let A be an $m\times n$ matrix and let $v_1, v_2, \ldots, v_k$ be vectors in $\mathbb{R}^n$. Show that if $Av_1, Av_2, \ldots, Av_k$ are linearly independent vectors in $\mathbb{R}^m$, then $v_1, v_2, \ldots, v_k$ are linearly independent in $\mathbb{R}^n$.
5. (a) Let A be an invertible $m\times m$ matrix and let $v_1, v_2, \ldots, v_k$ be vectors in $\mathbb{R}^m$. Show that if $v_1, v_2, \ldots, v_k$ are linearly independent vectors in $\mathbb{R}^m$, then $Av_1, Av_2, \ldots, Av_k$ are also linearly independent.
   (b) Find an example of a non-invertible $m\times m$ matrix B and linearly independent vectors $w_1, w_2, \ldots, w_k$ so that $Bw_1, Bw_2, \ldots, Bw_k$ are linearly dependent.
6. Find a basis for the solution space of the system:
$$\begin{cases} a+3b-c+2d &= 0\\ 2a+2b+c+2d &= 0\\ 4a \qquad +5c-7d &= 0\end{cases}$$
   What is the dimension of this subspace?
7. Find a basis for the solution space of the system:
$$\begin{cases} u+3v\ -w+2x+y &= 0\\ u+2v+4w+2x &= 0\\ 2u+8v\ +w+3x-y &= 0\end{cases}$$
   What is the dimension of this subspace?
8. $3x+2y-z=0$ is the equation of a plane $\mathcal{P}$ in $\mathbb{R}^3$ that passes through the origin, so $\mathcal{P}$ is a subspace of $\mathbb{R}^3$. Find a basis for $\mathcal{P}$. What is the dimension of this subspace?
9. Let $F = \begin{pmatrix} 1 & -2 & 1\\ 1 & 1 & -3\\ -1 & 8 & -9\end{pmatrix}$
   (a) Find a basis for the column space of F.
   (b) Find a basis for the column space of F'.
   (c) Is $(1,1,1)$ in the column space of F?
10. Let $A = \begin{pmatrix} 1 & -1 & 1 & 0 & -2\\ 2 & 0 & 1 & 1 & -1\\ 1 & -3 & 2 & -1 & -5\end{pmatrix}$
   (a) Find a basis for the column space of A.
   (b) Find a basis for the null space of A.
   (c) Is $(1,1,1)$ in the column space of A?
11. Let $B = \begin{pmatrix} 1 & 1 & 1\\ -1 & 1 & -3\\ 3 & 1 & 5\\ 2 & -1 & 5\end{pmatrix}$
   (a) Find a basis for the column space of B.
   (b) Find a basis for the column space of B'.
   (c) Is $(1,3,-1,-4)$ in the column space of B?
12. (a) Show that the vectors $(1,2,1)$, $(-1,1,0)$, and $(2,0,0)$ are a basis for $\mathbb{R}^3$.
   (b) Find the coordinates of $(0,0,1)$ with respect to this basis.
13. (a) The vectors $u=(2,1,-1)$, $v=(1,1,0)$, and $w=(1,2,1)$ are linearly dependent in $\mathbb{R}^3$.
   (b) Write $z=u+v-2w=(1,-2,-3)$ as a linear combination of two or fewer of these vectors.

14. Find a basis for the subspace of $\mathbb{R}^4$ spanned by $(1,-1,5,-5)$, $(1,1,-1,1)$, and $(2,1,1,-1)$. What is the dimension of this subspace?

15. Find a basis for the subspace of $\mathbb{R}^4$ spanned by $(1,-1,1,2)$, $(1,1,-1,1)$, and $(2,1,1,-1)$. What is the dimension of this subspace?

For each of the following matrices, find bases for the range and the nullspace, and find the dimensions of these subspaces.

16. $$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & -1 & 4 & 0 \end{pmatrix}$$

17. $$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & -1 & 1 \\ 3 & 3 & -1 & 2 \end{pmatrix}$$