Homework 14

1. Find the eigenvalues of the matrix A below, and find an orthonormal basis for $\mathbb{R}^2$ consisting of eigenvectors for A. What property of A guarantees that this is possible?

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$$

2. Let $B = \begin{pmatrix} 1 & 4 & -2 \\ 4 & 1 & 2 \\ -2 & 2 & 4 \end{pmatrix}$

   The characteristic polynomial of B is $p(\lambda) = -(\lambda - 5)^2(\lambda + 4)$. What are the eigenvalues of B? Find an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors for B.

3. The numbers -3, 1, and 5 are the eigenvalues of the matrix

$$E = \begin{pmatrix} 0 & -1 & -3 & 1 \\ -1 & 0 & 1 & -3 \\ -3 & 1 & 0 & -1 \\ 1 & -3 & -1 & 0 \end{pmatrix}$$

   Find an orthonormal basis for $\mathbb{R}^4$ that consists of eigenvectors for E.

4. The 5×5 matrix S is Hermitian (self–adjoint) and v is an eigenvector for S with eigenvalue -3. The vector w is perpendicular to v. Prove that Sw is also perpendicular to v.

5. The five-tuples $(2, 2, 1, -1, 1)$ and $(1, 1, 2, -1, -1)$ are both solutions of the system:

$$\begin{cases} a + b + 4c + d + e &= 8 \\ a - b + 2c + 2d + e &= 1 \\ 2a + b - c - d - 2e &= 4 \\ b + 3c + d + e &= 5 \\ 2a - b + c + 3d &= 0 \end{cases}$$

   (a) Without using Gaussian elimination or a machine, write down two non-trivial solutions of the associated homogeneous system.
   (b) Write down two other solutions of the given system.

6. In answering a question on her linear algebra homework, April claimed that the subspace $\mathcal{W}$ is spanned by the set $u_1 = (1, 0, 1)$ and $u_2 = (0, 1, -1)$. Michelle claimed that the subspace $\mathcal{W}$ is spanned by $v_1 = (1, 1, 0)$, $v_2 = (2, 1, 1)$, and $v_3 = (1, -1, 2)$. Do their answers agree with each other, that is, is the subspace spanned by the set $\{u_1, u_2\}$ the same as the subspace spanned by $\{v_1, v_2, v_3\}$?

7. In answering a question on his linear algebra homework, Max claimed that the subspace $\mathcal{U}$ is spanned by the set $u_1 = (1, 0, 1, 1)$, $u_2 = (0, 1, -1, 0)$ and $u_3 = (0, 0, 1, 2)$. Spike claimed that the subspace $\mathcal{U}$ is spanned by $v_1 = (1, 1, 0, 1)$, $v_2 = (2, 1, 1, 2)$, and $v_3 = (1, -1, 2, 1)$. Do they agree with each other, that is, is the subspace spanned by the set $\{u_1, u_2, u_3\}$ the same as the subspace spanned by $\{v_1, v_2, v_3\}$?

8. In doing his linear algebra homework, Eduardo found that if $u = (2, 1, -3)$, $v = (1, -1, 2)$, and $w = (1, 5, -12)$, then $2u - 3v - w = 0$. Does this calculation show that u, v, and w form a linearly dependent set, that they form a linearly independent set, or does it not show either of these?

9. In doing her linear algebra homework, Pauline was given that $p = (1, 1, -3)$, $q = (2, -1, 2)$, and $r = (1, -2, 5)$. She noticed that $0p + 0q + 0r = 0$. Does this observation show that p, q, and r form a linearly dependent set, that they form a linearly independent set, or does it not show either of these?

10. In doing their linear algebra homework, Abbie and John were given that $x = (2, -1, 1)$, $y = (1, 0, 2)$, and $z = (1, 1, -3)$. They found that the system $c_1x + c_2y + c_3z = 0$ has only one solution, namely, $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$. Does this calculation show that u, v, and w form a linearly dependent set, that they form a linearly independent set, or does it not show either of these?

11. Let A be an $m \times n$ matrix and let $v_1, v_2, \ldots, v_k$ be vectors in $\mathbb{R}^n$. Show that if $Av_1, Av_2, \ldots, Av_k$ are linearly independent vectors in $\mathbb{R}^m$, then $v_1, v_2, \ldots, v_k$ are linearly independent in $\mathbb{R}^n$.

12. (a) Let A be an invertible $m \times m$ matrix and let $v_1, v_2, \ldots, v_k$ be vectors in $\mathbb{R}^m$. Show that if $v_1, v_2, \ldots, v_k$ are linearly independent vectors in $\mathbb{R}^m$, then $Av_1, Av_2, \ldots, Av_k$ are also linearly independent.
    (b) Find an example of a non-invertible $m \times m$ matrix B and linearly independent vectors $w_1, w_2, \ldots, w_k$ so that $Bw_1, Bw_2, \ldots, Bw_k$ are linearly dependent.

13. Look over Homework 8......

14. Let $v_1 = (1, 1, 2, 0, 2)$, $v_2 = (1, -1, 1, 1, -1)$, $v_3 = (1, 2, -1, 1, 3)$, and $v_4 = (2, -1, -1, 3, -1)$ be vectors in $\mathbb{R}^5$.
    (a) How can you tell, without lifting a pencil or using a computer, that there is a non-zero vector that is perpendicular to each of v_1, v_2, v_3, and v_4?
    (b) Find a non-zero vector w in $\mathbb{R}^5$ that is perpendicular to each of v_1, v_2, v_3, and v_4.