Math 276: Solution to Problem 34b, page 160

Problem: Find a one-to-one correspondence between the natural numbers and the set of (positive) integers that are divisible by 5 but not divisible by 7.

Construction:

The first observation is that an integer m is a multiple of 5 if $m = 5n$ for some integer n, and it is also a multiple of 7 if and only if n is a multiple of 7. More specifically, n is a multiple of 7, say $n = 7k$, then $m = 5n = 5(7k) = 7(5k)$ so that $m = 5n$ is also a multiple of 7. On the other hand, if n is not a multiple of 7, then $m = 5n$ is not a multiple of 7:

$$\begin{aligned} m = 5n &= 5(7k+1) = 5(7k) + 5 = 7(5k) + 5 \\ m = 5n &= 5(7k+2) = 5(7k) + 10 = 7(5k+1) + 3 \\ m = 5n &= 5(7k+3) = 5(7k) + 15 = 7(5k+2) + 1 \\ m = 5n &= 5(7k+4) = 5(7k) + 20 = 7(5k+2) + 6 \\ m = 5n &= 5(7k+5) = 5(7k) + 25 = 7(5k+3) + 4 \\ m = 5n &= 5(7k+6) = 5(7k) + 30 = 7(5k+4) + 2 \end{aligned}$$

For each of the cases in which n is not divisible by 7, we see that $m = 5n$ is also not divisible by 7 because the above shows that m is 7 times an integer plus one of the remainders 1, 2, 3, 4, 5, or 6.

Thus, the integers we want to identify come from the 6 different remainders that give n not divisible by 7. We can pair these integers with the set of natural numbers by thinking about whether they are, or are not divisible by 6:

For p a positive integer,

$$f(p) = \begin{cases} 5(7k+1) & \text{if } p = 6k+1 \\ 5(7k+2) & \text{if } p = 6k+2 \\ 5(7k+3) & \text{if } p = 6k+3 \\ 5(7k+4) & \text{if } p = 6k+4 \\ 5(7k+5) & \text{if } p = 6k+5 \\ 5(7k-1) & \text{if } p = 6k \end{cases}$$

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To check this, we can see $f(1) = 5$ (because $p = 1 = 6 \cdot 0 + 1$), $f(2) = 10$ (because $p = 2 = 6 \cdot 0 + 2$), $f(3) = 15$ (because $p = 3 = 6 \cdot 0 + 3$), $f(4) = 20$ (because $p = 4 = 6 \cdot 0 + 4$), $f(5) = 25$ (because $p = 5 = 6 \cdot 0 + 5$), $f(6) = 30$ (because $p = 6 = 6 \cdot 1 + 0$), $f(7) = 40$ (because $p = 7 = 6 \cdot 1 + 1$), $f(8) = 45$ (because $p = 8 = 6 \cdot 1 + 2$), $f(9) = 50$ (because $p = 9 = 6 \cdot 1 + 3$), $f(10) = 55$ (because $p = 10 = 6 \cdot 1 + 4$), $f(11) = 60$ (because $p = 11 = 6 \cdot 1 + 5$), $f(12) = 65$ (because $p = 12 = 6 \cdot 2 + 0$), $f(13) = 75$ (because $p = 13 = 6 \cdot 2 + 1$), etc.