FINITE BLASCHKE PRODUCTS AS COMPOSITIONS OF OTHER FINITE BLASCHKE PRODUCTS

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Abstract. These notes answer the question "When can a finite Blaschke product B be written as a composition of two finite Blaschke products B_1 and B_2, that is, $B = B_1 \circ B_2$, in a non-trivial way, that is, where the order of each is greater than 1." It is shown that a group can be computed from B and its local inverses, and that compositional factorizations correspond to normal subgroups of this group. This manuscript was written in 1974 but not published because it was pointed out to the author that this was primarily a reconstruction of work of Ritt from 1922 and 1923, who reported on work on polynomials. It is being made public now because of recent interest in this subject by several mathematicians interested in different aspects of the problem and interested in applying these ideas to complex analysis and operator theory.

1. Introduction

From the point of view of these notes, for a positive integer n, a Blaschke product of order n (or n-fold Blaschke product) is an n-to-one analytic map of the open unit disk, $\mathbb{D}$, onto itself. It is well known that such maps are rational functions of order n, so have continuous extensions to the closed unit disk and the Riemann sphere that are n-to-one maps of these sets onto themselves, and have the form

$$B(z) = \lambda \prod_{k=1}^{n} \frac{z - \alpha_j}{1 - \overline{\alpha_j} z}$$

for $\alpha_1, \alpha_2, \cdots, \alpha_n$ points of the unit disk and $|\lambda| = 1$.

These notes were my first formal mathematical writing, developed at the beginning of the work on my thesis, and were written as a present to my former teacher Professor John Yarnelle on the occasion of his retirement from Hanover College, Hanover, Indiana, where I had been a student. The only original of these notes was given to Professor Yarnelle (since deceased) in December 1974 and what is presented here is a scan of the Xerox$^{\text{TM}}$ copy I made for myself at that time. These notes have never been formally circulated, but they have been shared over the years with several people and form the basis of the work in my thesis [2], especially in Section 2, my further work on commutants of analytic Toeplitz operators then [3, 4, 5], and more recently in work on multiple valued composition operators [6] and a return to questions of commutants of analytic multiplication operators [7]. In addition, they have formed the basis of my talk "An Unexpected Group", given to several undergraduate audiences in recent years starting in 2007 at Wabash College. In the past few years, more interest has been shown in this topic and it seems appropriate to make these ideas public and available to others who are working with related topics. Examples of a revival of interest of Ritt's ideas are in the work of R. G. Douglas and D. Zheng and their collaborators, for example [9, 10], and in the purely function theoretic questions such as the very nice work of Rickards [12] on decomposition of polynomials and the paper of Beardon and Ng [1].

The reason this is the first time these notes are being circulated is simple. In the fall of 1976, I gave a talk on this work in the analysis seminar at the University of Illinois at Urbana-Champaign where I was most junior of postdocs. The audience received it politely, and possibly with some interest, so as the end of the talk neared, I was feeling good at my first foray into departmental life. Then, at question time, the

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very distinguished Professor Joe Doob asked, "Didn't Ritt [13, 14] do something like this in the 1920's?" I was devastated and embarrassed and promptly put the manuscript in a drawer, thinking it unpublishable. In retrospect, I probably should have gone to Professor Doob for advice and written it up for publication with appropriate citations. Because Ritt's first work was on compositional factorization of polynomials, it is somewhat different than this, but it is obvious that the ideas involved apply to polynomials, Blaschke products, or rational functions more generally. I believe that many analysts today, as I was then, are ignorant of Ritt's work in this arena and, at the very least, his work deserves to be better known.

The remainder of this document is the scan of the original work for Professor Yarnelle from Fall 1974, a short addendum from a year later that describes the application of these ideas to factorization of an analytic map on the disk into a composition of an analytic function and a finite Blaschke product, and a short bibliography of some work related to these ideas.

In the original notes, given a (normalized) finite Blaschke product I of order n, a group G_I is described as a permutation group of the branches of the local inverses of the Blaschke product I acted on by loops (based at 0) in a subset of the disk, the disk with $n(n-1)$ points removed. The main theorem of the notes is the following.

Theorem 3.1. *Let I be a finite Blaschke product normalized as above.*

If $\mathcal{P}$ is a partition of the set of branches of I^{-1} at 0, $\{g_1, g_2, \cdots, g_n\}$, that G_I respects, then there are finite Blaschke products $J_{\mathcal{P}}$ and $b_{\mathcal{P}}$ with the order of $b_{\mathcal{P}}$ the same as the order of $\mathcal{P}$ so that

$$I = J_{\mathcal{P}} \circ b_{\mathcal{P}}$$

Conversely, if J and b are finite Blaschke products so that $I = J \circ b$, then there is a partition $\mathcal{P}_b$ of the set of branches of I^{-1} at 0 which G_I respects such that the order of $\mathcal{P}_b$ is the same as the order of b.

Moreover, if $\mathcal{P}$ and b are as above, then

$$\mathcal{P}_{b_{\mathcal{P}}} = \mathcal{P}_b \qquad \text{and} \qquad b_{\mathcal{P}_b} = b$$

It is shown that the compositional factorizations of G_I are associated with normal subgroups of G_I, but that the association is more complicated than one might hope in that non-trivial normal subgroups of G_I can be associated with trivial compositional factorizations of I. However, the association is strong enough, then if one knows all of the normal subgroups of G_I, then one can construct all possible non-trivial factorizations of I into compositions of finite Blaschke products and inequivalent factorizations of I as compositions correspond to different normal subgroups of G_I.

The main theorem of the addendum is the following.

Theorem. *If $f : \mathbb{D} \mapsto f(\mathbb{D})$ is analytic and exactly n-to-one [as a map of the open unit disk onto the image $f(\mathbb{D})$], then there is a finite Blaschke product ϕ and a one-to-one function $\widetilde{f}$ so that $f = \widetilde{f} \circ \phi$.*

This result has the obvious corollaries that $f(\mathbb{D})$ is simply connected and f' has exactly $n-1$ zeros in the disk.

1974

Finite Blaschke Products as Compositions of Other Finite Blaschke Products

by Carl Cowen

For Professor John Yarnelle, Hanover College on the occasion of his retirement.

0. __Introduction__: An n-fold Blaschke product is an n to 1 conformal map of the unit disk $D = \{z \in \mathbb{C} \mid |z| < 1\}$ onto itself. The composition of an n-fold and an m-fold Blaschke product is an $m \times n$-fold Blaschke product. This paper concerns discovering whether a given Blaschke product is the composition of two other Blaschke products in a non-trivial way (obviously $b = id \circ b = b \circ id$). The problem is solved by associating the Blaschke product with a finite group (the group of covering transformations of the Riemann surface of the inverse of the Blaschke product) in such a way that compositions correspond to normal subgroups of the group. The main theorem is in section 3, as well as some examples.

Section 4 treats the problem of finding common compositions, that is, if $\{\Psi_\alpha\}$ are maps of D analytically into $\mathbb{C}$, and b is a finite Blaschke product, we find a finite Blaschke product J so that $b = \tilde{b} \circ J$ and $\Psi_\alpha = \tilde{\Psi}_\alpha \circ J$, and $\tilde{b}$ and the $\tilde{\Psi}_\alpha$ have only trivial common compositions.

1. __Terminology and definitions__: For convenience, and to avoid trivial cases we assume that if I is a Blaschke product, then $I(0) = 0$, $I'(0) > 0$ and if $I(\alpha) = 0$ then $I'(\alpha) \neq 0$. Then, if $0 = \alpha_1, \alpha_2, \ldots, \alpha_n$ are the zeros of I, we have

$$I(z) = z \prod_{k=2}^{n} \frac{\bar{\alpha}_k}{|\alpha_k|} \left(\frac{\alpha_k - z}{1 - \bar{\alpha}_k z} \right).$$

Here, $|\alpha_k|<1$ and the $\{\alpha_k\}_{k=1}^n$ are distinct. These assumptions are normalizing assumptions: if I is a finite Blaschke product, let $\beta \in D$ be a regular value of I and $\alpha \in D$ be such that $I(\alpha)=\beta$. Then $\tilde{I}(z) = \lambda \dfrac{\beta - I\left(\frac{\alpha - z}{1-\bar{\alpha} z}\right)}{1-\bar{\beta} I\left(\frac{\alpha-z}{1-\bar{\alpha}z}\right)}$ where λ is a suitable constant of modulus 1 satisfies the normalizing assumptions, and clearly any statement about writing $\tilde{I}$ as a compocition implies a similar statement about I and vice versa.

Given the finite Blaschke product I, let S be the set of critical values of I, that is $S = \{w \in D \mid w = I(z) \text{ and } I'(z)=0\}$. It is easily seen that S is a finite set, in fact that S has at most $n-1$ points where n is the number of zeroes of I. Let $\tilde{S} = I(S)$. Clearly $\tilde{S}$ is also a finite set, in fact $\tilde{S}$ has at most $n(n-1)$ points. We consider the n-valued analytic function I^{-1} which is defined and arbitrarily continuable in $D-S$. (We will use later that I^{-1} is arbitrarily continueble in $U-S$ where U is a neighborhood of $\bar{D}$, but the argument can be restated to avoid this.)
Since $0 \in D-S$ I^{-1} has n branches at 0, say $g_1, g_2 \cdots g_n$ where $g_1(0)=0$.

Suppose γ is a curve in $D-S$ so that $\gamma(0)=\gamma(1)=0$. Then g_1 can be continued along γ, and we will denote the final element of this continuation by $\gamma^* g_1$. $\gamma^* g_1$ is a branch of I^{-1} at 0, so $\gamma^* g_1 \in \{g_1, g_2 \cdots g_n\}$. $\gamma^* g_i$, $i=2, \cdots n$ are defined analogously and we see that γ^* is a permutation of the set $\{g_1, \cdots g_n\}$. If γ and δ are two loops at 0 in $D-S$, and $\gamma\delta$ is the loop at 0 in $D-S$ defined in the usual way, it is clear that $(\gamma\delta)^* = \gamma^* \delta^*$. It is a consequence of the homotopy lemma in the theory of analytic

Continuation that if γ and δ are homotopic, that $\gamma^* = \delta^*$.

<u>Definition 1.1</u> If I is a finite Blaschke product, let G_I, <u>the group associated with I</u>, be the set of permutations on $\{g_1, \dots g_n\}$ induced by loops at 0 in $D-S$, i.e. $G_I = \{\gamma^* \mid \gamma : [0,1] \to D-S, \gamma(0) = \gamma(1) = 0\}$. So G_I is a quotient of $\pi_1(D-S)$ and is isomorphic to a subgroup of S_n.

We will need a few definitions and lemmas about groups (like G_I) acting on sets (like $\{g_1, g_2, \dots g_n\}$).

<u>Definition 1.2</u> Let G be a group which acts transitively on a set X, and let $\mathcal{P}$ be a partition of X. We will say <u>G respects $\mathcal{P}$</u> if for each $g \in G$ and each $P \in \mathcal{P}$, there is $P' \in \mathcal{P}$ such that $gP \subset P'$.

Now $g^{-1}P' \subset P''$ if G respects $\mathcal{P}$, but clearly $g^{-1}P' \cap P \neq \emptyset$, so actually $g^{-1}P' \subset P$, and we see that $gP = P'$. In particular, if G respects $\mathcal{P}$, each element of $\mathcal{P}$ has the same cardinality, and we will call this cardinality the <u>order of $\mathcal{P}$</u>.

<u>Lemma 1.3</u> Let G be a group which acts transitively on a set X. If $\mathcal{P}$ is a partition of X which G respects, then $H = \{h \in G \mid hP = P$ for all $P \in \mathcal{P}\}$ is a normal subgroup of G.

Conversely, if H is a normal subgroup of G, then the orbit space of H, i.e. $\{Hx \mid x \in X\}$, is a partition of X which G respects.

<u>Proof:</u> Suppose $h \in H = \{h \in G \mid hP = P$ for all $P \in \mathcal{P}\}$, and suppose $g \in G$. For $P \in \mathcal{P}$, $gP \in \mathcal{P}$, so $(g^{-1}hg)P = (g^{-1}h)(gP) = g^{-1}(h(gP)) = g^{-1}(gP) = P$ and $g^{-1}hg \in H$.

If $g \in G$ and H is normal in G, then $g(Hx) = (gH)x = (Hg)x = H(gx)$. Thus g respects the partition $\{Hx \mid x \in X\}$.
Q.E.D.

Now it is reasonably clear that the normal subgroup arising from the partition $\{Hx \mid x \in X\}$ where H is normal, is H.

On the other hand, if $\mathcal{P}$ is a partition, and $H = \{h \mid hP = P, \text{ all } P \in \mathcal{P}\}$, it is not necessarily the case that $\{Hx \mid x \in X\} = \mathcal{P}$. For example, let $X = \{-1, 1\} \times \{1, 2, 3\}$ and $G = S_3$, where the action of G on X is $\sigma(a,b) = ((\operatorname{sign} \sigma)a, \sigma b)$. Then G respects the partition $\mathcal{P} = \{\{(-1,1),(1,1)\}; \{(-1,2),(1,2)\}, \{(-1,3),(1,3)\}\}$ and acts transitively on X, but the only element of G which leaves each member of $\mathcal{P}$ fixed is the identity. We will see below, Corollary 2.3 together with section 3, that this cannot happen in cases of interest to us.

It should be noted that in his book Modern Algebra, B. L. van der Waerden calls a group acting on a set with a partition a system of imprimitivity.

2. Theorems about the groups, G_I, and examples:

Proposition 2.1: Suppose I, J, and b are finite Blaschke products such that $I = J \circ b$. Then the map $\pi : G_I \to G_J$, defined by letting $\pi(\gamma_I^*)$ be the element of G_J induced by γ, where γ_I^* is the element of G_I induced by γ, is a homomorphism of G_I onto G_J. We will write $\pi(\gamma_I^*) = \gamma_J^*$.

Proof: Let S_I and S_J be the critical values of I and J respectively. We see by the chain rule that $S_J \subset S_I$, therefore any curve admissible in defining G_I is also admissible in defining G_J. Moreover, each element of J^{-1} at 0 can be expressed (in several ways) as $b \circ g_i$ where g_i is a branch of I^{-1} at 0. Then by permanence of functional relations, if γ is a curve in $D - S_I$, continuing $b \circ g_i$ along γ as a branch of J^{-1} is the same as composing b with g_i continued along γ as a branch of I^{-1}. That is $\gamma_J^*(b \circ g_i) = b \circ \gamma_I^* g_i$. Thus if γ and δ are two curves in $D - S_I$ with $\gamma_I^* = \delta_I^*$, then $\gamma_J^* = b \circ \gamma_I^* = b \circ \delta_I^* = \delta_J^*$, so π is well defined. It is now obvious that π is a homomorphism.

Now let γ be a curve in $D - S_J$ with $\gamma(0) = \gamma(1) = 0$. Since $S_I - S_J$ is a finite set, there is a curve δ in $D - S_I$ homotopic to γ relative to $D - S_J$. Therefore $\delta_J^* = \gamma_J^*$. Since $\delta_J^* = \pi(\delta_I^*)$, π is onto G_J. Q.E.D.

Proposition 2.2: If I is a finite Blaschke product of order n, i.e. I is an n to 1 map of D onto D, then G_I has an element of order n.

Proof: We recall that for some neighborhood U of $\overline{D}$, I^{-1} is arbitrarily continuable in $U-S$ and that I maps ∂D onto ∂D, n to 1, with $I'(\lambda) \neq 0$ for $|\lambda|=1$.

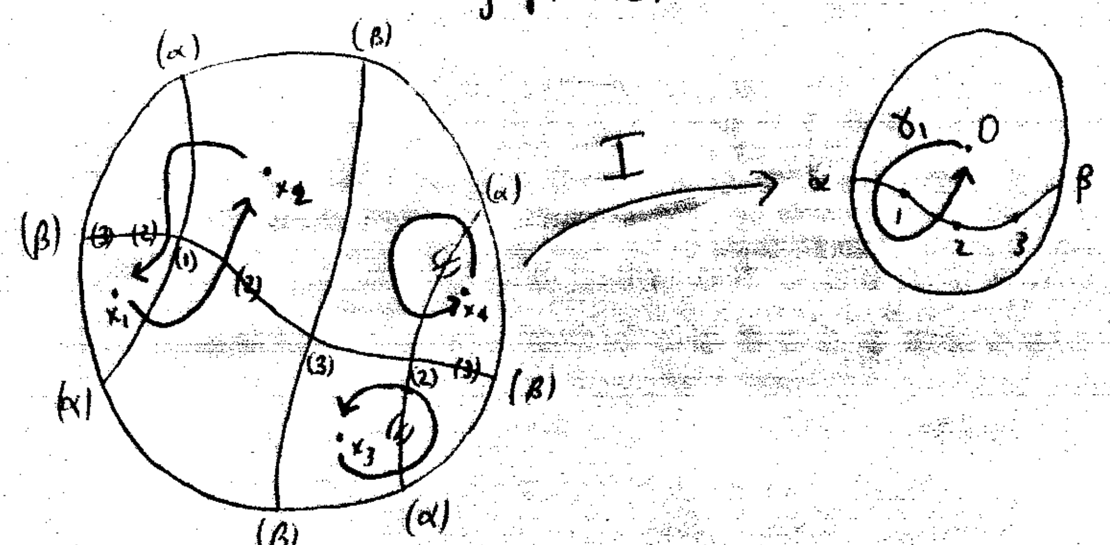
Let δ be a path in $D-S$ with $\delta(0)=0$, $\delta(1)=1$. Choose g_1 a branch of I^{-1} at 0 and continue g_1 along δ to a branch $\delta^* g_1$ of I^{-1} at 1, denoting $\delta^* g_1(1)$ by $e^{i\theta_1}$. Let $I^{-1}(1)$ be $e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, \dots, e^{i\theta_n}$ where $\theta_1 < \theta_2 < \theta_3 < \dots < \theta_n < \theta_1 + 2\pi$. Let $\tilde{\delta}$ be the path $\tilde{\delta}(t) = e^{2\pi i t}$, and let $\gamma = \delta^{-1}\tilde{\delta}\delta$, that is γ is a loop at 0 which connects 0 to 1 wraps counterclockwise around ∂D and connects 1 to 0 again. From the definition of $e^{i\theta_2}$, we see that $(\tilde{\delta}\delta)^* g_1(1) = e^{i\theta_2}$; $(\tilde{\delta}^2\delta)^* g_1(1) = e^{i\theta_3}$, etc, so that $(\tilde{\delta}^r\delta)^* g_1(1) = (\tilde{\delta}^{r'}\delta)^* g_1(1)$ if and only if $r \equiv r' \bmod n$. Therefore, $\gamma^{*r} g_1 = \gamma^{*r'} g_1$ if and only if $r \equiv r' \bmod n$, and by the same reasoning, if g_j is any branch of I^{-1} at 0, $\gamma^{*r} g_j = \gamma^{*r'} g_j$ if and only if $r \equiv r' \bmod n$. Therefore γ^* is of order n. Since S is at some positive distance from ∂D, there is a loop $\tilde{\gamma}$ at 0 such that γ and $\tilde{\gamma}$ are homotopic in $U-S$ and such that $\tilde{\gamma}$ is in $D-S$. Therefore $\tilde{\gamma}^* = \gamma^*$ and the above constructed permutation is actually in G_I. Q.E.D.

Corollary 2.3: Suppose I, J and b are as in Proposition 2.1 and suppose h is the branch of J^{-1} at 0 with $h(0)=0$ and $g_1 \cdots g_k$ are the branches of I^{-1} so that $b \circ g_i = h$ for $i=1, \dots k$. Then $\ker \pi$ acts transitively on $g_1, \dots g_k$.

Proof: From the hypotheses the order of b is k, so the order of J is n/k, and the proof of Prop. 2.2 shows that if γ^* is as above $\pi(\gamma^*)$ has order n/k. The proof also shows that g_1, $(\gamma^{*\, n/k}) g_1$, $(\gamma^{*\, \frac{2n}{k}}) g_1, \dots (\gamma^{*\, n-n/k}) g_1$ are distinct and $h = b \circ g_1 = b \circ (\gamma^{*\, n/k}) g_1 = \dots = b \circ (\gamma^{*\, n-n/k}) g_1$ so this must be the set $g_1, \dots, g_k$. Q.E.D.

We will now compute the groups, G_I, for some specific finite Blaschke products. The idea is to draw a "picture" of the Blaschke product, and to compute the group from the picture. First we choose a pair of points α and β on ∂D, and a simple curve, δ, joining α to β passing through the critical values of I, not passing through 0. Now $I^{-1}(\delta)$ will locally be a curve, except at points of D for which I' vanishes, in which case $I^{-1}(\delta)$ will be intersecting curves. δ divides D into two domains, one of which contains 0. $I^{-1}(\delta)$ divides D into $2n$ domains n of which contain a point of $I^{-1}(0)$. The inverse images of the critical points of I will all lie on $I^{-1}(\delta)$ and I will preserve the order along $I^{-1}(\delta)$. $\pi_1(D-S)$ has at most $n-1$ generators, so G_I will also. The permutations that each of these generators induces can be be found by noting the places where the curves cross δ.

<u>Example 2.4</u>: Let $\tilde{I}(z) = z^2\left(\frac{z-1/3}{1-\frac{1}{3}z}\right)\left(\frac{z-1/2}{1-\frac{1}{2}z}\right)$. Let $I(z)$ be a normalization of $\tilde{I}$ as per section 1. Now I is a Blaschke product of order 4 with 3 distinct critical values, denoted in the picture by 1, 2, 3. In the picture $I^{-1}(\alpha)$ will be denoted by (α); $I^{-1}(\beta)$ denoted by (β); etc. and $I^{-1}(0)$ denoted by x_1, x_2, x_3, x_4. For an appropriate curve δ we have the following picture:



The picture can be drawn almost entirely using the fact that traversing δ means starting at α passing through 1, 2, 3 ending at β and keeping 0 on the left. So in the inverse image, starting at (α) passing through (1), (2), (3) and ending at (β) keeping x_i on the left. One of the generators of $\pi(D-S)$ has been drawn in: γ_1 a loop starting at 0, crossing δ between α and 1, and crossing back between 1 and 2. So inverse images of γ_1 must start at x_i cross $I^{-1}(\delta)$ between (α) and (1) and again between (1) and (2) ending at x_j. It is clear that if g_1, g_2, g_3 and g_4 are the branches of I^{-1} with $g_i(0) = x_i$ then

$$\gamma_1^*(g_1) = g_2 \ ; \ \gamma_1^*(g_2) = g_1 \ ; \ \gamma_1^*(g_3) = g_3 \ ; \text{ and } \gamma_1^*(g_4) = g_4 .$$

Letting γ_2 and γ_3 be loops at 0 passing between 1 and 2 then 2 and 3; and between 2 and 3 then 3 and β we see that

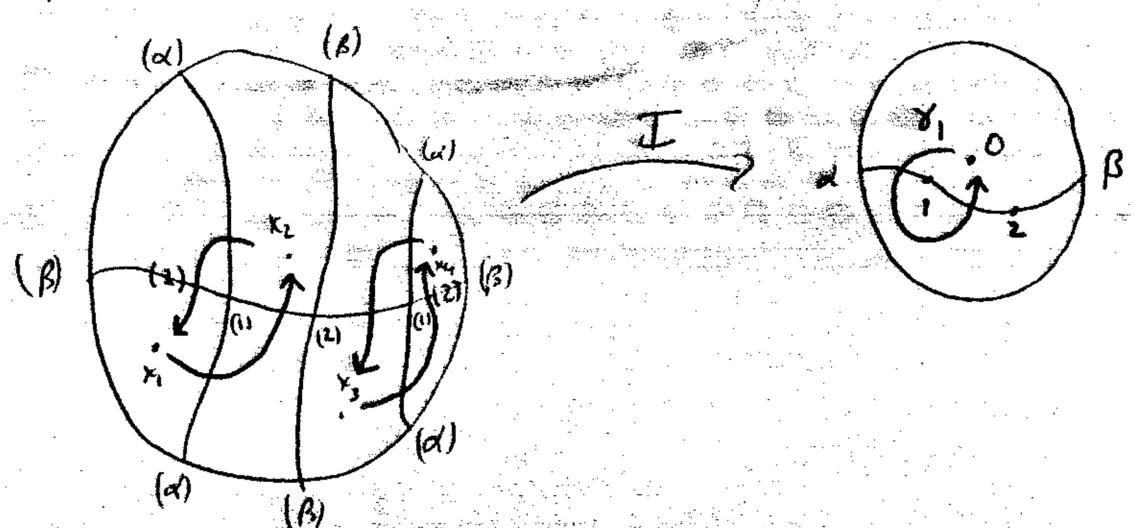
$$\gamma_2^*(g_1) = g_1 \ ; \ \gamma_2^*(g_2) = g_2 \ ; \ \gamma_2^*(g_3) = g_4 \ ; \text{ and } \gamma_2^*(g_4) = g_3$$

$$\gamma_3^*(g_1) = g_1 \ ; \ \gamma_3^*(g_2) = g_3 \ ; \ \gamma_3^*(g_3) = g_2 \ ; \text{ and } \gamma_3^*(g_4) = g_4$$

Thus G_I is isomorphic to S_4 under the isomorphism η:

$$\eta(\gamma_1^*) = (12) \qquad \eta(\gamma_2^*) = (34) \qquad \text{and} \qquad \eta(\gamma_3^*) = (23).$$

<u>Example 2.5</u> Let $\tilde{I}(z) = z^2\left(\frac{z-1/2}{1-1/2 z}\right)^2$ and let I be a normalization of $\tilde{I}$. Now I has only two critical values and we have the following picture:



With γ_1 as pictured, and γ_2 defined to be a loop at 0 passing first between 1 and 2 then between 2 and β, we have:

$\gamma_1^*(g_1) = g_2$; $\gamma_1^*(g_2) = g_1$; $\gamma_1^*(g_3) = g_4$; and $\gamma_1^*(g_4) = g_3$;

$\gamma_2^*(g_1) = g_1$; $\gamma_2^*(g_2) = g_3$; $\gamma_2^*(g_3) = g_2$; and $\gamma_2^*(g_4) = g_4$.

This group is isomorphic to the dihedral group with 8 elements, D_4.

3. The Main Theorem

Theorem 3.1 Let I be a finite Blaschke product normalized as above.

If $\mathcal{P}$ is a partition of the set of branches of I^{-1} at 0, $\{g_1, \dots g_n\}$, which G_I respects, then the are finite Blaschke products $J_{\mathcal{P}}$ and $b_{\mathcal{P}}$, with the order of $b_{\mathcal{P}}$ the same as the order of $\mathcal{P}$, so that

$$I = J_{\mathcal{P}} \circ b_{\mathcal{P}}.$$

Converseley, if J and b are finite Blaschke products so that $I = J \circ b$, then there is a partition $\mathcal{P}_b$ of the set of branches of I^{-1} at 0 which G_I respects such that the order of $\mathcal{P}_b$ is the same as the order of b.

Moreover, if $\mathcal{P}$ and b are as above,

$$\mathcal{P}_{b_{\mathcal{P}}} = \mathcal{P} \qquad \text{and} \qquad b_{\mathcal{P}_b} = b.$$

Proof: Renumbering if necessary, we assume that the partition $\mathcal{P}$ is $P_1 = \{g_1, g_2, \dots g_k\}$; $P_2 = \{g_{k+1}, g_{k+2}, \dots g_{2k}\}$; $\dots$ $P_m = \{g_{(m-1)k+1} \dots g_{mk}\}$ where $mk = n$. [That such a numbering is possible follows from the fact that G_I respects $\mathcal{P}$ and the comment following definition 1.2.] Assume also $g_1(0) = 0$.

Now $g_j(I(z))$ is arbitrarily continuable in $D - \tilde{S}_I$ since $I(D - \tilde{S}_I) \subset D - S_I$ and g_j is arbitrarily continuable in $D - S_I$. For z in some neighborhood of 0 in $D - \tilde{S}_I$ define

$$b_{\mathcal{P}}(z) = z \prod_{j=2}^{k} \frac{\overline{g_j(0)}}{|g_j(0)|} g_j(I(z)) = g_1(I(z)) \prod_{j=2} \frac{\overline{g_j(0)}}{|g_j(0)|} g_j(I(z))$$

(this latter equality hold because $g_1(I(z)) = z$, which follows from the fact that $I(g_1(I(z))) = I(z)$ and $g_1(I(0)) = 0$.)

Since each $g_1(I(z))$ is arbitrarily continuable in $D - \tilde{S}_I$, so is $b_\wp$, with the appropriate product formula holding for all branches of $b_\wp$ in $D - \tilde{S}_I$.

Now suppose δ is a loop at 0 in $D - \tilde{S}_I$. Continuing $b_\wp$ along δ is continuing $g_1(w) \prod_{j=2}^{k} \frac{\overline{g_j(0)}}{|g_j(0)|} g_j(w)$ along the curve $\gamma = I \circ \delta$ in $D - S_I$. Clearly $\gamma^*(g_1) = g_1$, for $g_1(I(z)) = z$ is single valued in $\mathbb{C}$, and continuing g_1 along γ is continuing z along δ. But since G_I respects the partition $\wp$, we must have $\gamma^* P_1 = P_1$. This means that continuing $b_\wp$ along δ only changes the order of the factors in the product, so that continuing $b_\wp$ yields $b_\wp$. Since $b_\wp$ is arbitrarily continuable in $D - \tilde{S}_I$ and single valued in a neighborhood of zero, $b_\wp$ must be single valued in all of $D - \tilde{S}_I$. Since $|b_\wp| < 1$ in $D - \tilde{S}_I$, and $\tilde{S}_I$ is a finite set, actually $b_\wp$ defines a holomorphic function on all of D.

In the same way, define $d_\wp(z) = \prod_{j=k+1}^{n} \frac{\overline{g_j(0)}}{|g_j(0)|} g_j(I(z))$. Then just as above we see that $d_\wp$ is analytic in all of D.

Now $|b_\wp(z)\, d_\wp(z)| = \prod_{j=1}^{n} |g_j(I(z))| = |I(z)|$, so that $b_\wp(z)\, d_\wp(z) = \lambda I(z)$ for some λ, $|\lambda| = 1$. [The latter equality can be found in a paper of R. McLaughlin, [ ] or since these are finite Blaschke products, proved easily.] Evaluating the derivative at 0, we see that $\lambda = 1$. Since $|b_\wp(z)| \leq 1$ and $|d_\wp(z)| < 1$

for all $z \in D$, we see that actually, b_ρ and d_ρ are finite Blaschke products. In particular, b_ρ is the Blaschke product with zeroes $0 = g_1(0), g_2(0), \cdots g_k(0)$ and $b'_\rho(0) > 0$, so that the order of b_ρ is the same as the order of ρ.

We claim that for $r = 0, 1, \ldots, m-1$, that $b_\rho(g_{rk+1}(0)) = b_\rho(g_{rk+2}(0)) = \cdots = b_\rho(g_{r(k+1)}(0))$. To see this, if r is as above and $1 \le j \le k$, let γ be a loop at 0 in $D - S_I$ so that $\gamma^* g_1 = g_{rk+j}$. Let δ be the lifting of γ to $D - \tilde{S}_I$ with $\delta(0) = 0$; thus $\delta(1) = g_{rk+j}(0)$. Then by definition $b_\rho(g_{rk+j}(0))$ is the continuation of $z \prod_{\ell=2}^{k} \frac{\overline{g_\ell(0)}}{|g_\ell(0)|} g_\ell(I(z))$ along δ. Since $\gamma^* g_1 = g_{rk+j}$ and G_I respects ρ, we see that $\gamma^* P_1 = P_{r+1}$ and that
$$b_\rho(g_{rk+j}(0)) = \left(\prod_{\ell=2}^{k} \frac{\overline{g_\ell(0)}}{|g_\ell(0)|}\right)\left(\prod_{\ell=1}^{k} g_{rk+\ell}(0)\right),$$
which does not depend on j.

Now let J_ρ be the finite Blaschke product with $J'_\rho(0) > 0$ and zeroes $b_\rho(g_1(0)) = 0$; $b_\rho(g_{k+1}(0))$; $b_\rho(g_{2k+1}(0))$; $\cdots$ $b_\rho(g_{(m-1)k+1}(0))$. Thus $J_\rho \circ b_\rho$ is a finite Blaschke product with zeroes $g_1(0), g_2(0), \cdots, g_n(0)$; and $(J_\rho \circ b_\rho)'(0) > 0$. But I is a finite Blaschke product with these zeroes and $I'(0) > 0$, so $I = J_\rho \circ b_\rho$.

To prove the converse, let $I = J \circ b$, and let $g_1, \cdots g_n$ be the branches of I^{-1} at 0. For each g_j, $b \circ g_j$ is a branch of J^{-1} at 0, and we define a partition ρ_b by the equivalence relation, $g_j \sim g_{j'}$ if $b \circ g_j = b \circ g_{j'}$ on their common domain. That G_I respects ρ_b is just permanence of functional relations. Suppose $g_1(0) = 0$, then the order of $\mathcal{P}_b$ is just the number of g_i such that $b \circ g_i = b \circ g_1$, which is the same as the number of g_i such that $b(g_i(0)) = b(g_1(0)) = 0$, since by the normalization I and J have distinct zeros. That is, the order of $\mathcal{P}_b$ is the same as the order of b.

That $\mathcal{P}_{b_{\mathcal{P}}} = \mathcal{P}$ and $b_{\mathcal{P}_b} = b$ now follows from the definitions of the Blaschke products and the partitions. Q.E.D.

<u>Corollary 3.2</u> Let I be a finite Blaschke product as above.

If H is a normal subgroup of G_I, there are finite Blaschke products, J_H and b_H so that $I = J_H \circ b_H$, namely $b_H = b_{\mathcal{P}}$ where $\mathcal{P} = \{Hg\}$ where g runs over the branches of I^{-1} at 0.

Conversely, if J and b are finite Blaschke products so that $I = J \circ b$, there is a normal subgroup, H_b, of G_I, namely $H_b = \ker \pi$ where $\pi : G_I \to G_J$ is the homomorphism of proposition 2.1.

Moreover, $b_{H_b} = b$ and $H_{b_H} = \{ \gamma^* \in G_I \mid \gamma^* g \in Hg$ for all g branches of I^{-1} at $0\}$, so $H_{b_H} \supset H$.

<u>Proof</u>: Only the last two statements need proof, and the equality $b_{H_b} = b$ is just a restatement of Corollary 2.3.

Now $\gamma^* \in \ker \pi$ if and only if $b_H \circ (\gamma^* g) = b_H \circ g$ for all branches, g, of I^{-1} at 0. One the other hand, from the definition of b_H this holds if and only if $\gamma^* g \in Hg$, for all g branches of I^{-1} at 0. Q.E.D.

Naturally, we want to avoid the trivial compositions $I = z \circ I$ and $I = I \circ (z)$. Theorem 3.1 says that we can do that by avoiding the trivial partitions $\mathcal{P} = \{\{g_1, \ldots g_n\}\}$ and $\mathcal{P} = \{\{g_1\}, \{g_2\}, \ldots \{g_n\}\}$.

It is, of course, easier to work with the normal subgroups of G_I, but unfortunately, two normal subgroups may give rise to the same partition, so to the same decomposition. In particular, some normal subgroups give the partition $P = \{\{g_1, \cdots g_n\}\}$. However, from the definition of G_I as a set of different functions, the only normal subgroups which gives the other trivial partition is $\{e\}$. So we have:

Corollary 3.3 If H is a normal subgroup of G_I such that the order of H is strictly less than the order of I, then I has a non-trivial decomposition, $I = J_H \circ b_H$.

Proof: The order of the partition induced by H is less than or equal to the order of H, but is not 1. Thus the order of b_H is bigger than 1, less than order I. Q.E.D.

3.4 Continuation of example 2.4 In example 2.4, we determined the group G_I for the given Blaschke product to be S_4. Now S_4 has only 4 normal subgroups: S_4, A_4, $\{e\}$, $\{e, (12)(34), (13)(24), (14)(23)\}$. Since each of these is transitive, except $\{e\}$, the partitions are all trivial so we conclude that the given finite Blaschke product cannot be written as a non-trivial composition of other finite Blaschke products.

3.5 Continuation of example 2.5 In example 2.5, we determined the group G_I for the given Blaschke product to be D_4. Now D_4 has six normal subgroups: $\{e\}$; D_4; $C = \{e, \gamma_1^*\gamma_2^*\gamma_1^*\gamma_2^*\}$ the center;

$N_1 = \{e, \gamma_1^*\gamma_2^*\gamma_1^*\gamma_2^*, \gamma_2^*, \gamma_1^*\gamma_2^*\gamma_1^*\}$;

$N_2 = \{e, \gamma_1^*\gamma_2^*\gamma_1^*\gamma_2^*, \gamma_1^*, \gamma_2^*\gamma_1^*\gamma_2^*\}$;

and $N_3 = \{e, \gamma_1^*\gamma_2^*\gamma_1^*\gamma_2^*, \gamma_1^*\gamma_2^*, \gamma_2^*\gamma_1^*\}$; where γ_1^* and γ_2^* are as in example 2.5.

Checking these subgroups we find that D_4, N_2, N_3 are transitive, so $\{e\}, D_4, N_2, N_3$ yield trivial decompositions of I, but C and N_1 are not transitive, and both give the partition $\mathcal{P} = \{\{1,4\},\{2,3\}\}$. Therefore I has exactly one decomposition in a non-trivial way. [This decomposition was obvious from the way in which was presented: $\tilde{I} = \left[z\left(\frac{z-1/2}{1-1/2 z}\right)\right]^2$.]

We close this section with a theorem which further illustrates the connections between the partitions and compositions.

<u>Theorem 3.6</u> If I is a finite Blaschke product, normalized as above, and $\mathcal{P}$ and $\mathcal{P}'$ are partitions of the set of branches of I^{-1} at 0 such that $\mathcal{P}'$ is finer than $\mathcal{P}$, and both $\mathcal{P}$ and $\mathcal{P}'$ are respected by G_I, then there is a finite Blaschke product ϕ so that $b_{\mathcal{P}} = \phi \circ b_{\mathcal{P}'}$, and therefore $J_{\mathcal{P}'} = J_{\mathcal{P}} \circ \phi$.

<u>Proof</u>: We follow the proof of Theorem 3.1, paying more attention to the partitions. We number the branches of I^{-1} at 0 so that $\mathcal{P}' = \{\{g_1, g_2 \cdots g_\ell\}, \{g_{\ell+1}, \cdots g_{2\ell}\}, \cdots \}$ and so that $\mathcal{P} = \{\{g_1, g_2, \cdots g_k\}, \{g_{k+1}, \cdots g_{2k}\}, \cdots\}$ where $\{g_1, \cdots g_k\} = \{g_1 \cdots g_\ell\} \cup \{g_{\ell+1} \cdots g_{2\ell}\}$ $\cup \cdots \cup \{g_{(r-1)\ell+1}, \cdots g_{r\ell}\}$ where $r\ell = k$, and $g_1(0) = 0$.

Then $b_{\mathcal{P}}(z) = z \prod_{j=2}^{k} \frac{\overline{g_j(0)}}{|g_j(0)|} g_j(I(z))$ and $b_{\mathcal{P}'} = z \prod_{j=2}^{\ell} \frac{\overline{g_j(0)}}{|g_j(0)|} g_j(I(z))$.

As in the proof of 3.1 we show $b_{\mathcal{P}'}(g_{s\ell+1}(0)) = b_{\mathcal{P}'}(g_{s\ell+2}(0)) = \cdots = b_{\mathcal{P}'}(g_{\ldots}$

Then let $\phi(z)$ be the finite Blaschke product with zeroes at $0 = g_1(0)$, $g_{\ell+1}(0), \cdots, g_{(r-1)\ell+1}(0)$ and prove as in 3.1 that $b_{\mathcal{P}} = \phi \circ b_{\mathcal{P}'}$.

Q.E.D.

Theorem 3.6 really says that if I is a finite Blaschke product, normalized as above, then the set of finite Blaschke products $\{ b \mid \text{there is } J \text{ a finite Blaschke product with } I = J \circ b \}$ has a complete lattice structure under composition, where we say $b_1 \succ b_2$ if there is ϕ a finite Blaschke product and $b_1 = \phi \circ b_2$.
This lattice is just the lattice of partitions of the branches of I^{-1} at 0 which G_I respects, which is obviously a complete lattice under the relation "finer".

4. Common compositions for a finite Blaschke product and a set of holomorphic functions.

Theorem 4.1 Let $\mathcal{F} = \{\psi_\gamma\}_{\gamma \in T}$ be a family of holomorphic functions, $\psi_\gamma : D \to \mathbb{C}$ and let I be a finite Blaschke product, normalized as above.

Then there are finite Blaschke products $b_{\mathcal{F}}$ and $J_{\mathcal{F}}$ and functions $\tilde{\psi}_\gamma : D \to \mathbb{C}$ so that $I = J_{\mathcal{F}} \circ b_{\mathcal{F}}$ and $\psi_\gamma = \tilde{\psi}_\gamma \circ b_{\mathcal{F}}$ for all $\gamma \in T$. Moreover $b_{\mathcal{F}}$ is the unique, maximal finite Blaschke product with this property, in the sense that if b^* and J^* have the given properties, then there is a finite Blaschke product, ϕ, with $b_{\mathcal{F}} = \phi \circ b^*$.

Proof: Let $g_1, \ldots g_n$ be the branches of I^{-1} at 0.
Let $\mathcal{P}$ be the partition induced by the equivalence relation $g_i \sim g_j$ if $\psi_\gamma \circ g_i = \psi_\gamma \circ g_j$ on their common domain, for all $\gamma \in T$. G_I respects the partition $\mathcal{P}$ by the principle of permanence of functional relations.
By perhaps renumbering, we assume $g_1(0) = 0$ and that
$\mathcal{P} = \{ \{g_1 \ldots g_k\}, \{g_{k+1}, \ldots g_{2k}\}, \ldots, \{g_{n-k+1}, \ldots g_n\} \}$.

We claim that $b_{\mathcal{F}} = b_{\mathcal{P}}$ and $J_{\mathcal{F}} = J_{\mathcal{P}}$ have the required properties.

For $J \in T$ and for z in a suitable neighborhood of 0, let $\tilde{\Psi}_J(z) = \Psi_J \circ g_1 \circ J_{\#}(z)$. Then $\tilde{\Psi}_J$ is arbitrarily continuable in $D - J_{\#}^{-1}(S_I)$. Suppose δ is a loop at 0 in $D - J_{\#}^{-1}(S_I)$. Continuing $\tilde{\Psi}_J$ along is the same as continuing $\Psi_J \circ g_1$ along $J_{\#}(\delta)$. But continuing g_1 along $J_{\#}(\delta)$ gives $g_j \in \{g_1, \dots g_k\}$ so continuing $\Psi_J \circ g_1$ gives $\Psi_J \circ g_j = \Psi_J \circ g_1$. Therefore $\tilde{\Psi}_J$ is single valued in $D - J_{\#}^{-1}(S_I)$. Since $\tilde{\Psi}_J$ is bounded in some neighborhood of each point of $J_{\#}^{-1}(S_I)$, $\tilde{\Psi}_J$ is actually defined and holomorphic in all of D. Moreover, $\tilde{\Psi}_J \circ b_{\#} = \Psi_J \circ g_1 \circ J_{\#} \circ b_{\#} = \Psi_J \circ g_1 \circ I = \Psi_J$.

Now suppose $I = J^* \circ b^*$ is another decomposition of I with $\Psi_J = \Psi_J^* \circ b^*$ for each $J \in T$. Let $\mathcal{P}'$ be the partition of $\{g_1 \dots g_n\}$, induced by the equivalence relation $g_i \approx g_j$ if $b^* \circ g_i = b^* \circ g_j$. G_I respects $\mathcal{P}'$ by the permanence of functional relations, and also $\mathcal{P}'$ is finer than $\mathcal{P}$, for if $b^* \circ g_i = b^* \circ g_j$ then

$$\Psi_J \circ g_i = \Psi_J^*(b^* g_i) = \Psi_J^*(b^* g_j) = \Psi_J \circ g_j .$$

Now theorem 3.6 says $b_{\#} = \Phi \circ b^*$ for some finite Blaschke product Φ.

Q.E.D

5. Conclusion

In the preceeding discussion, we have found a description of the ways in which a finite Blaschke product can be written as a composition. This study is motivated in part by problems involving the commutant of the Toeplitz operator T_I, studied for example by Deddens and Wong, [ ], or Thompson, [ ]. In particular, the results of section 4 can be used to characterize the operators which commute with T_I and T_Ψ for $\Psi \in H^\infty$, (Thompson [ ]). All of these questions make sense in case I is an infinite Blaschke product, and it is hoped that similar theorems can be proved for the more complex case. Indeed, some limited results are known for that case which suggests that such a generalization is possible.

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<u>Theorem</u> If $f : D \to f(D)$ is analytic and exactly n to 1, then there is a finite Blaschke product ϕ, with n zeroes, and a one-to-one function $\tilde{f}$, so that $f = \tilde{f} \circ \phi$.

<u>Corollary</u> $f(D)$ is simply connected.

<u>Corollary</u> f' has exactly $n-1$ zeroes in D.

[<u>Note</u>: The motivation for this proof comes from two observations. First, if ψ is a finite Blaschke product, $\psi(0) = 0$, and $\eta_1, \dots \eta_n$ are the branches of ψ^{-1}, then there is a constant λ, $|\lambda| = 1$ so that $\psi(z) = \lambda \prod_{j=1}^{n} \eta_j(\psi(z))$. Second, if $f = \tilde{f} \circ \phi$ where $\tilde{f}$ is one-to-one, then $f^{-1} \circ f = \phi^{-1} \circ \phi$. To prove the theorem, we construct the Blaschke product as such a product.]

<u>Proof</u>: Let $F = \{ f(z) : f'(z) = 0 \}$. F is countable since $f'(z) \not\equiv 0$. We claim F is closed. Given $w \in f(D) \setminus F$, let $z_1, z_2, \dots z_n$ be the points in D so that $f(z_j) = w$. Since $w \notin F$, $f'(z_j) \neq 0$ for $j = 1, \dots, n$, and the z_j are distinct. For each j, $j = 1, 2, \dots n$, choose U_j a neighborhood of z_j so that $f'(z) \neq 0$ for $z \in U_j$ and $U_j \cap U_k = \emptyset$, if $j \neq k$. $W = \bigcap_{j=1}^{n} f(U_j)$ is a neighborhood of w, and $W \cap F = \emptyset$. Thus $f(D) \setminus F$ is open, and F is closed. Since F is countable, $f(D) \setminus F$ is a domain.

With out loss of generality, we assume $f(0) \in f(D) \setminus F$.

Let $g_1, g_2, \ldots, g_n$ be the branches of f^{-1} at $f(0)$. Each g_j is arbitrarily continuable in $f(D) \setminus F$. We define $\phi(z) = \prod_{j=1}^{n} g_j(f(z))$ in a neighborhood of 0. Now ϕ is arbitrarily continuable in $D \setminus f^{-1}(F)$, and moreover, ϕ is single valued. For if ϕ is continued along some loop at 0 in $D \setminus f^{-1}(F)$, at worst, the order of the factors is changed. Since ϕ is single valued, arbitrarily continuable, and bounded by 1 in $D \setminus f^{-1}(F)$, and since $f^{-1}(F)$ is a closed, countable set ϕ can be extended to a holomorphic function in D.

We claim that if K is compact in $f(D)$, then $f^{-1}(K)$ is compact in D. Suppose $\{a_k\}_{k=1}^{\infty} \subset f^{-1}(K)$. Then $\{f(a_k)\} \subset K$, and without loss of generality we may assume $f(a_k) \to w_0$, $w_0 \in K$. Let $f^{-1}(w_0) = \{z_1, z_2, \ldots, z_r\}$ with multiplicities $m_1, \ldots, m_r$ respectively $\left(\sum_{j=1}^{r} m_j = n\right)$. Now $\{a_k\}_{k=1}^{\infty}$ clusters at one of the points z_j, for if not, there are disjoint neighborhoods $V_1, V_2, \ldots, V_r$ of $z_1, \ldots, z_r$ respectively so that $f : V_j \to f(V_j)$ is m_j to 1 and $\{a_k\}_{k=1}^{\infty} \cap \left(\bigcup_{j=1}^{r} V_j\right) = \emptyset$. But this is impossible since $f(a_k)$ is eventually in $\bigcap_{j=1}^{r} f(V_j)$, and $f^{-1}\left(\bigcap_{j=1}^{r} f(V_j)\right) \subset \bigcup_{j=1}^{r} V_j$. So some subsequence of $\{a_k\}$ converges to z_j for some $j = 1, \ldots, r$, and $z_j \in f^{-1}(K)$.

We can now show that ϕ is a finite Blaschke product by showing that $|\phi(z)| \to 1$ uniformly as $|z| \to 1$. For $r < 1$, let $\overline{D}_r = \{z \mid |z| \le r\}$. $\overline{D}_r$ is compact, so $f(\overline{D}_r)$ is compact, and by the above $f^{-1}(f(\overline{D}_r))$ is compact. So given $r < 1$, there is s, $0 < s < 1$ with $z \notin f^{-1}(f(\overline{D}_r))$ if $|z| > s$. Hence if $|z| > s$, $|\phi(z)| \geq r^n$. Moreover, $\phi(z) = 0$ if and only if $f(z) = f(0)$, so ϕ has exactly n zeros.

From the product formula for ϕ, we see that if $f(z) = f(z')$ then $\phi(z) = \phi(z')$, but since this accounts for all n times ϕ takes the value $\phi(z)$, we see that if $\phi(z) = \phi(z')$ then $f(z) = f(z')$. We now define $\tilde{f}(z) = f(\phi^{-1}(z))$. The above shows that $\tilde{f}$ is well defined, and $f = \tilde{f} \circ \phi$. Q.E.D.

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