UNITARY EQUIVALENCE OF ONE-PARAMETER GROUPS OF TOEPLITZ AND COMPOSITION OPERATORS

CARL C. COWEN AND EVA A. GALLARDO-GUTIÉRREZ

Abstract. The composition operators on H^2 whose symbols are hyperbolic automorphisms of the unit disk fixing ± 1 comprise a one-parameter group and the analytic Toeplitz operators coming from covering maps of annuli centered at the origin whose radii are reciprocals also form a one-parameter group. Using the eigenvectors of the composition operators and of the adjoints of the Toeplitz operators, a direct unitary equivalence is found between the restrictions to zH^2 of the group of adjoints of these composition operators and the group of Toeplitz operators. On the other hand, it is shown that there is not a unitary equivalence of the groups of adjoints of the composition operators and the Toeplitz operators on the whole of H^2.

1. Introduction

Our goal in this note is to explore equivalence between the adjoints of the composition operators on H^2 coming from the hyperbolic automorphisms of the disk with fixed points at ± 1 and the usual analytic Toeplitz operators associated with covering maps of annuli centered at the origin whose radii are reciprocals. The unitary equivalence we find is on zH^2, not on H^2. In fact, we show (Theorem 8) that there is a unitary operator on zH^2 that gives a unitary equivalence of the group of adjoints of the composition operators and the group of analytic Toeplitz operators. In addition, we show (Theorem 4) that there is *not* a unitary equivalence between these groups on all of H^2.

Our approach is to look at the eigenspaces of the adjoints of the Toeplitz operators and the eigenspaces of the composition operators, or more precisely, of the infinitesimal generators of their groups because these are one-dimensional, and construct a unitary equivalence of these operators directly by using their eigenfunctions. As a corollary, since the compressed adjoint Toeplitz operators and the compressed composition operators are unitarily equivalent, the restricted (to zH^2) Toeplitz operators and the restricted adjoint composition operators are also unitarily equivalent.

Understanding the structure of adjoints of composition operators as multiplications by analytic functions, especially relating to subnormality, extends

Date: 11 July 2010.

2010 *Mathematics Subject Classification.* Primary: 47B38, Secondary: 47B20, 47B33, 47B35.

Key words and phrases. Composition operator, Toeplitz operator, subnormal operator, unitary equivalence.

This work was partially supported by Plan Nacional I+D grant no. MTM2007-61446 and Gobierno de Aragón research group *Análisis Matemático y Aplicaciones*, ref. DGA E-64. The first author thanks the Área de Análisis Matemático de la Universidad de Zaragoza for the hospitality shown him during summer 2009 that made this work possible.

back to the first few years these operators were studied. Subnormality of the Cesàro operator on H^2 was shown by Kriete and Trutt [12] in 1971 and Deddens [7] showed in 1972 that the adjoints of some composition operators on H^2 are analytic functions of the Cesáro operator, so are subnormal. The connections between subnormality and composition operators were extended by Nordgren, Rosenthal, and Wintrobe [14], Cowen [4], and finally Cowen and Kriete [5] by giving proofs showing that the class of composition operators with subnormal adjoints on H^2 includes some composition operators whose symbols are hyperbolic linear fractional maps of the disk into itself, in particular, it includes the hyperbolic automorphisms with fixed points at ± 1. However, in contrast to our direct approach in this note, in [14] and [4], the subnormality was established indirectly through moment criteria, which does not usually tie closely to the structure of the operators as multiplications by analytic functions. In [5], like in [12], measures were constructed to show that the operators studied are restrictions to $P^2(\mu)$ of the normal operator of multiplication by a bounded analytic function on $L^2(\mu)$.

2. Comparison of the Adjoint Toeplitz and Composition Groups

We wish to find a relationship between the Toeplitz operator whose symbol is the covering map of the disk onto an annulus and the composition operator whose symbol is a hyperbolic automorphism of the disk with fixed points ± 1. We believe (hope?) this might be true because the Toeplitz operator is subnormal and the composition operator has subnormal adjoint and the composition operator and the adjoint of the Toeplitz operator both have point spectra that are open annuli, each eigenvalue having infinite multiplicity. Rather than use an approach that uses measures or moments, as in [5], we try a novel approach that uses their groups.

Consider the set (for $-\infty < t < \infty$) of composition operators with symbols C_{φ_t} where

$$\varphi_t(z) = \frac{(1+e^{-t})z + (1-e^{-t})}{(1-e^{-t})z + (1+e^{-t})}$$

which has fixed points at 1 and -1 and derivatives at these fixed points e^{-t} and e^t. Thus, for $t > 0$, the Denjoy-Wolff point of φ_t is 1. Easy computations show that this is a one-parameter group of operators, $C_{\varphi_t} C_{\varphi_s} = C_{\varphi_{s+t}}$, and it is not too difficult to see that this group is strongly continuous.

Let us compute the infinitesimal generator of this group:

$$\begin{aligned}(Hf)(z) &= \left(\frac{d}{dt}\bigg|_{t=0} C_{\varphi_t} f \right)(z) = \frac{d}{dt}\bigg|_{t=0} f(\varphi_t(z)) \\ &= f'(\varphi_t(z)) \frac{2(1+e^{-t})e^{-t}(1-z^2)}{[(1-e^{-t})z + (1+e{-}t)]^2}\bigg|_{t=0} \\ &= f'(z)\frac{1-z^2}{2}\end{aligned}$$

(We note, but will not prove because our focus will be on the eigenfunctions, that H is a closed operator with domain $\{f \in H^2 : f'(z)(1-z^2) \in H^2\}$.)

We want to find the eigenvalues of this differential operator that correspond to eigenvectors that are in H^2. That is, we want to solve the differential equation

$f'(z)(1-z^2)/2 = \lambda f$ and choose those λ that correspond to f in H^2. We get

$$\frac{1}{f}\,df = \lambda\frac{2}{1-z^2}\,dz = \lambda\left(\frac{1}{1+z}+\frac{1}{1-z}\right)\,dz$$

which has solutions (up to an additive constant)

$$\log f = \lambda\left(\log(1+z)-\log(1-z)\right) = \log\left(\frac{1+z}{1-z}\right)^{\lambda}$$

The functions w_λ are eigenvectors of the infinitesimal generator corresponding to λ

$$w_\lambda(z) = f(z) = \left(\frac{1+z}{1-z}\right)^{\lambda} = \left(\frac{1-z}{1+z}\right)^{-\lambda}$$

and they are in H^2 for $-1/2 < \operatorname{Re}\lambda < 1/2$. Note that the eigenspaces of the infinitesimal generator are one-dimensional! Use of the theory of semigroups or a direct computation from the expressions for φ_t and w_λ shows that $C_{\varphi_t}w_\lambda = e^{\lambda t}w_\lambda$. In particular, for $t>0$, the point spectrum of C_{φ_t} is

$$\sigma_p(C_{\varphi_t}) = \{\lambda : e^{-t/2} < |\lambda| < e^{t/2}\}$$

Let us now consider the Toeplitz operators whose symbols are maps of the disk onto annuli centered at the origin with radii that are reciprocals of each other, that is, the same family of annuli as occur above as the spectra of the composition operators. Such maps are (for $s>0$)

$$g(z) = e^{si\log\left(\frac{1-z}{1+z}\right)} = \left(\frac{1-z}{1+z}\right)^{si}$$

and the point spectrum of the adjoint of the Toeplitz operator is

$$g(\mathbb{D}) = \{\zeta : e^{-\pi s/2} < |\zeta| < e^{\pi s/2}\}$$

Since our goal is to match up with the group above, we choose a normalization so that $t=1$ corresponds to the annulus $\{\zeta : e^{-1/2} < |\zeta| < e^{1/2}\}$, and we let

$$\psi_t(z) = e^{\left(\frac{ti}{\pi}\log\left(\frac{1-z}{1+z}\right)\right)} = \left(\frac{1-z}{1+z}\right)^{\frac{ti}{\pi}}$$

These Toeplitz operators also form a strongly continuous group, $T_{\psi_t}T_{\psi_s} = T_{\psi_{s+t}}$, and we want to find the infinitesimal generator. For h in H^2,

$$\begin{aligned}(Gh)(z) &= \left(\frac{d}{dt}\bigg|_{t=0}(T_{\psi_t}h)\right)(z) = \frac{d}{dt}\bigg|_{t=0} e^{\frac{ti}{\pi}\log\left(\frac{1-z}{1+z}\right)}h(z)\\ &= \left(\frac{i}{\pi}\log\left(\frac{1-z}{1+z}\right)e^{\frac{ti}{\pi}\log\left(\frac{1-z}{1+z}\right)}\right)\bigg|_{t=0} h(z)\\ &= \frac{i}{\pi}\log\left(\frac{1-z}{1+z}\right)h(z)\end{aligned}$$

That is, the infinitesimal generator, G, of the group is an (unbounded) analytic Toeplitz operator. As is well known, the kernel functions for evaluation at α in the disk, $K_\alpha(z) = (1-\overline{\alpha}z)^{-1}$, are eigenvectors for adjoints of analytic Toeplitz operators

$$G^*K_\alpha = -\frac{i}{\pi}\log\left(\frac{1-\overline{\alpha}}{1+\overline{\alpha}}\right)K_\alpha$$

and we see, also in this case, that the eigenspaces are one dimensional. We also have

$$T_{\psi_t}^* K_\alpha = \overline{\psi_t(\alpha)} K_\alpha = \left(\frac{1-\overline{\alpha}}{1+\overline{\alpha}}\right)^{-\frac{ti}{\pi}} K_\alpha$$

Now, we need to get the relationship between α and λ to compare eigenfunctions for the same eigenvalue. Using $t = 1$, we have

$$e^\lambda = \left(\frac{1-\overline{\alpha}}{1+\overline{\alpha}}\right)^{-\frac{i}{\pi}}$$

$$e^{-\frac{\pi\lambda}{i}} = e^{i\pi\lambda} = \frac{1-\overline{\alpha}}{1+\overline{\alpha}}$$

so finally,

$$\overline{\alpha} = \frac{1-e^{i\pi\lambda}}{1+e^{i\pi\lambda}} = \frac{e^{-i\pi\lambda/2}-e^{i\pi\lambda/2}}{e^{-i\pi\lambda/2}+e^{i\pi\lambda/2}} = \frac{-i\sin\left(\lambda\frac{\pi}{2}\right)}{\cos\left(\lambda\frac{\pi}{2}\right)}$$

We will match up the eigenspaces for the infinitesimal generators of the two groups to try to obtain the equivalence we are looking for.

For $-1/2 < \operatorname{Re}\lambda < 1/2$, let

$$w_\lambda = \left(\frac{1-z}{1+z}\right)^{-\lambda} \qquad \text{and} \qquad v_\lambda = \left(1 - \frac{-i\sin\left(\lambda\frac{\pi}{2}\right)}{\cos\left(\lambda\frac{\pi}{2}\right)} z\right)^{-1}$$

which are the eigenvectors found above that correspond to the eigenvalue λ for the infinitesimal generators for the two groups. Of course, we are aware that we have made natural, but ultimately arbitrary, choices of eigenvectors for the two cases. We hope that these choices will suggest an isomorphism of the space that will connect the composition and the Toeplitz operators. The following lemma says that at least we have enough vectors in each case to use linear combinations of eigenfunctions to get close to every vector in the space.

Lemma 1. *Let w_λ and v_λ be as above. Then the span of $\{v_\lambda : -1/2 < \lambda < 1/2\}$ and the span of $\{w_\lambda : -1/2 < \lambda < 1/2\}$ are each dense in H^2.*

Proof. The v_λ are just kernels for point evaluations for functions in H^2. If f in H^2 is perpendicular to each v_λ then, for $\alpha = i\sin(\lambda\pi/2)/\cos(\lambda\pi/2)$ as above, we have $0 = \langle f, v_\lambda\rangle = f(\alpha)$, and this is true for each α in the intersection of the unit disk with the imaginary axis. Since f is analytic in the disk, we must have $f = 0$. This means the span of $\{v_\lambda : -1/2 < \lambda < 1/2\}$ is dense in H^2.

For the second half, let us prove that the span of $\{w_\lambda : 0 < \lambda < 1/2\}$ is dense in H^2, from which, clearly, the statement of the lemma will follow. Let us consider the natural surjective isometry from H^2 onto $L^2(0,\infty)$ (see [15], for instance). This means that it is enough to prove that the linear span of the functions in $L^2(0,\infty)$ given by

$$e_\lambda(x) = \int_0^x t^{\lambda-1} e^{-(x-t)}\, dt, \qquad (x > 0)$$

for $0 < \lambda < 1/2$ is dense in $L^2(0,\infty)$. In order to show this, assume f in $L^2(0,\infty)$ satisfies

$$\int_0^\infty e_\lambda(x)\,\overline{f(x)}\, dx = 0$$

for any $0 < \lambda < 1/2$. The Fubini theorem yields

$$\int_0^\infty t^{\lambda-1} \left(\int_t^\infty e^{-(x-t)}\, \overline{f(x)}\, dx \right) \, dt = 0. \tag{1}$$

Let us denote by F the $L^2(0,\infty)$ function

$$F(t) = \int_t^\infty e^{-(x-t)}\, \overline{f(x)}\, dx, \qquad (t \in (0,\infty))$$

Note that expression (1) is related to the Mellin Transform of F, defined for those $w \in \mathbb{C}$ such that the integral

$$\widetilde{F}(w) = \int_0^\infty t^{w-1} F(t)\, dt,$$

converges. Since $\widetilde{F}$ is an analytic map on the fundamental strip of convergence $\{0 < \operatorname{Re} w < 1/2\}$ and from expression (1) one gets $\widetilde{F}(\lambda) = 0$ for every $0 < \lambda < 1/2$, one deduces that $\widetilde{F} = 0$. Hence, $F(t) = 0$ for any $t \in (0,\infty)$. From here, it follows that $f = 0$, proving the statement of the lemma. □

Corollary 2. *If M is any subspace of H^2 and P is the orthogonal projection of H^2 onto M, then the span of $\{Pv_\lambda : -1/2 < \lambda < 1/2\}$ and the span of $\{Pw_\lambda : -1/2 < \lambda < 1/2\}$ are each dense in M.*

Proof. Suppose u is a vector in H^2 and x is a vector in M. Because $\|P\| = 1$, we have

$$\|x - Pu\| = \|Px - Pu\| = \|P(x-u)\| \le \|x - u\|$$

In particular, for every linear combination of the v_λ's or the w_λ's, the distance between x and the linear combination is no less than the distance between x and the projection of the linear combination, so the density of the projections in M follows from the density of the linear combinations in H^2. □

If there actually is an isomorphism, then the internal relationships between the vectors w_λ for different λ must have a strong connection with the same relationships for the v_λ. Thus, we will compute both $\langle w_\lambda, w_\mu \rangle$ and $\langle v_\lambda, v_\mu \rangle$ for λ and μ in the strip, indeed, it should be sufficient to do so for just the real numbers $-1/2 < \lambda, \mu < 1/2$.

Because the vectors v_λ and v_μ are just kernel functions, their inner products are easy to calculate. Recalling that we are taking λ and μ real,

$$\begin{aligned}
\langle v_\lambda, v_\mu \rangle &= \left(1 - \left(\frac{-i \sin(\frac{\pi}{2}\lambda)}{\cos(\frac{\pi}{2}\lambda)} \right) \left(\frac{i \sin(\frac{\pi}{2}\mu)}{\cos(\frac{\pi}{2}\mu)} \right) \right)^{-1} \\
&= \frac{\cos(\frac{\pi}{2}\lambda)\cos(\frac{\pi}{2}\mu)}{\cos(\frac{\pi}{2}\lambda)\cos(\frac{\pi}{2}\mu) - \sin(\frac{\pi}{2}\lambda)\sin(\frac{\pi}{2}\mu)} = \frac{\cos(\frac{\pi}{2}\lambda)\cos(\frac{\pi}{2}\mu)}{\cos(\frac{\pi}{2}(\lambda+\mu))}
\end{aligned}$$

The corresponding calculation for w_λ and w_μ is somewhat more difficult.

$$
\begin{aligned}
\langle w_\lambda, w_\mu \rangle &= \int_{-\pi}^{\pi} \left(\frac{1-e^{i\theta}}{1+e^{i\theta}}\right)^{-\lambda} \left(\frac{1-e^{-i\theta}}{1+e^{-i\theta}}\right)^{-\mu} \frac{d\theta}{2\pi} \\
&= \int_{0}^{\pi} \left(\frac{1-e^{i\theta}}{1+e^{i\theta}}\right)^{-\lambda} \left(\frac{1-e^{-i\theta}}{1+e^{-i\theta}}\right)^{-\mu} \frac{d\theta}{2\pi} \\
&\qquad + \int_{-\pi}^{0} \left(\frac{1-e^{i\theta}}{1+e^{i\theta}}\right)^{-\lambda} \left(\frac{1-e^{-i\theta}}{1+e^{-i\theta}}\right)^{-\mu} \frac{d\theta}{2\pi} \\
&= \int_{0}^{\pi} \left(\frac{1-e^{i\theta}}{1+e^{i\theta}}\right)^{-\lambda} \left(\frac{1-e^{-i\theta}}{1+e^{-i\theta}}\right)^{-\mu} \frac{d\theta}{2\pi} \\
&\qquad + \int_{0}^{\pi} \left(\frac{1-e^{-i\theta}}{1+e^{-i\theta}}\right)^{-\lambda} \left(\frac{1-e^{i\theta}}{1+e^{i\theta}}\right)^{-\mu} \frac{d\theta}{2\pi} \\
&= \int_{0}^{\pi} \left(\frac{1-e^{i\theta}}{1+e^{i\theta}}\right)^{-\lambda} \left(-\frac{1-e^{i\theta}}{1+e^{i\theta}}\right)^{-\mu} \frac{d\theta}{2\pi} \\
&\qquad + \int_{0}^{\pi} \left(-\frac{1-e^{i\theta}}{1+e^{i\theta}}\right)^{-\lambda} \left(\frac{1-e^{i\theta}}{1+e^{i\theta}}\right)^{-\mu} \frac{d\theta}{2\pi}
\end{aligned}
$$

Making a change of variables, by taking the disk to the half plane to replace the unit circle by the imaginary axis,

$$\frac{1-e^{i\theta}}{1+e^{i\theta}} = -ix$$

and integrating on the real line, we get

$$
\begin{aligned}
\langle w_\lambda, w_\mu \rangle &= \int_{0}^{\infty} \frac{(-ix)^{-\lambda}(ix)^{-\mu}}{1+x^2}\frac{dx}{\pi} + \int_{0}^{\infty} \frac{(ix)^{-\lambda}(-ix)^{-\mu}}{1+x^2}\frac{dx}{\pi} \\
&= \frac{(-i)^{-\lambda}i^{-\mu} + i^{-\lambda}(-i)^{-\mu}}{\pi} \int_{0}^{\infty} \frac{x^{-\lambda-\mu}}{1+x^2}\, dx \\
&= \frac{e^{i\frac{\pi}{2}\lambda}e^{-i\frac{\pi}{2}\mu} + e^{-i\frac{\pi}{2}\lambda}e^{i\frac{\pi}{2}\mu}}{\pi} \int_{0}^{\infty} \frac{x^{-\lambda-\mu}}{1+x^2}\, dx \\
&= \frac{2\cos(\frac{\pi}{2}(\lambda-\mu))}{\pi} \int_{0}^{\infty} \frac{x^{-\lambda-\mu}}{1+x^2}\, dx
\end{aligned}
$$

where we have used $\pm i = e^{\pm i\frac{\pi}{2}}$. A computation using a computer algebra system (for example [13]) or a standard table of integrals (for example [16, p. 423, #486]) gives

$$\int_{0}^{\infty} \frac{x^{-\lambda-\mu}}{1+x^2}\, dx = \frac{\pi}{2\cos(\frac{\pi}{2}(\lambda+\mu))}$$

so we get

$$\langle w_\lambda, w_\mu \rangle = \frac{2\cos(\frac{\pi}{2}(\lambda-\mu))}{\pi} \frac{\pi}{2\cos(\frac{\pi}{2}(\lambda+\mu))} = \frac{\cos(\frac{\pi}{2}(\lambda-\mu))}{\cos(\frac{\pi}{2}(\lambda+\mu))} \tag{2}$$

Comparing the formula above for $\langle v_\lambda, v_\mu\rangle$ with this result for $\langle w_\lambda, w_\mu\rangle$, we see that they are similar in form but not exactly equal.

$$\langle v_\lambda, v_\mu\rangle = \frac{\cos(\frac{\pi}{2}\lambda)\cos(\frac{\pi}{2}\mu)}{\cos(\frac{\pi}{2}(\lambda+\mu))} \tag{3}$$

However, consider the following computation

$$\begin{aligned} 2\langle v_\lambda, v_\mu\rangle &= \frac{2\cos(\frac{\pi}{2}\lambda)\cos(\frac{\pi}{2}\mu)}{\cos(\frac{\pi}{2}(\lambda+\mu))} \\ &= \frac{\cos(\frac{\pi}{2}\lambda)\cos(\frac{\pi}{2}\mu)+\sin(\frac{\pi}{2}\lambda)\sin(\frac{\pi}{2}\mu)}{\cos(\frac{\pi}{2}(\lambda+\mu))} + \frac{\cos(\frac{\pi}{2}\lambda)\cos(\frac{\pi}{2}\mu)-\sin(\frac{\pi}{2}\lambda)\sin(\frac{\pi}{2}\mu)}{\cos(\frac{\pi}{2}(\lambda+\mu))} \\ &= \frac{\cos(\frac{\pi}{2}(\lambda-\mu))}{\cos(\frac{\pi}{2}(\lambda+\mu))} + \frac{\cos(\frac{\pi}{2}(\lambda+\mu))}{\cos(\frac{\pi}{2}(\lambda+\mu))} = \langle w_\lambda, w_\mu\rangle + 1 \end{aligned} \tag{4}$$

If a unitary operator shows two operators are equivalent, the unitary operator must carry the eigenspaces of one onto the eigenspaces of the other. In our case, if G^* and H are unitarily equivalent, the unitary must carry the eigenspace spanned by v_λ onto the eigenspace spanned by w_λ. However, we will see Equations (2) and (3) are inconsistent with this relationship.

Lemma 3. *Let $\mathcal{H}$ be a Hilbert space, let u_1, v_1, u_2, and v_2 be non-zero vectors in $\mathcal{H}$, and let $M_1 = span\{u_1\}$, $N_1 = span\{v_1\}$, $M_2 = span\{u_2\}$, and $N_2 = span\{v_2\}$. There is a unitary operator U on $\mathcal{H}$ such that $UM_1 = M_2$ and $UN_1 = N_2$ if and only if*

$$\frac{|\langle u_1, v_1\rangle|}{\|u_1\|\|v_1\|} = \frac{|\langle u_2, v_2\rangle|}{\|u_2\|\|v_2\|}$$

Proof. ($\Rightarrow$) If U is unitary and $UM_1 = M_2$ and $UN_1 = N_2$, then there are complex numbers α and β so that $Uu_1 = \alpha u_2$ and $Uv_1 = \beta v_2$. Since unitary operators preserve inner products, we know that

$$\begin{aligned} \|u_1\| &= \|Uu_1\| = \|\alpha u_2\| = |\alpha|\|u_2\| \\ \|v_1\| &= \|Uv_1\| = \|\beta v_2\| = |\beta|\|v_2\| \\ |\langle u_1, v_1\rangle| &= |\langle Uu_1, Uv_1\rangle| = |\langle \alpha u_2, \beta v_2\rangle| = |\alpha||\beta||\langle u_2, v_2\rangle| \end{aligned}$$

Combining these three equalities, we get the desired conclusion:

$$\frac{|\langle u_1, v_1\rangle|}{\|u_1\|\|v_1\|} = \frac{|\alpha||\beta||\langle u_2, v_2\rangle|}{|\alpha\|u_2\|\|\beta|\|v_2\|} = \frac{|\langle u_2, v_2\rangle|}{\|u_2\|\|v_2\|}$$

($\Leftarrow$) On the other hand, suppose we have vectors as above so that

$$\frac{|\langle u_1, v_1\rangle|}{\|u_1\|\|v_1\|} = \frac{|\langle u_2, v_2\rangle|}{\|u_2\|\|v_2\|}$$

If we let $\alpha = \|u_1\|/\|u_2\|$ and $\gamma = \|v_1\|/\|v_2\|$, then this equality gives

$$|\langle \alpha u_2, \gamma v_2\rangle| = \alpha\gamma|\langle u_2, v_2\rangle| = \alpha\gamma\|u_2\|\|v_2\|\frac{|\langle u_1, v_1\rangle|}{\|u_1\|\|v_1\|} = |\langle u_1, v_1\rangle|$$

Since these absolute values agree, we can find β so that $|\beta| = \gamma$ and $\langle \alpha u_2, \beta v_2\rangle = \langle u_1, v_1\rangle$.

Since we now have

$$\begin{aligned}\langle u_1, u_1\rangle &= |\alpha|^2\langle u_2, u_2\rangle = \langle \alpha u_2, \alpha u_2\rangle \\ \langle v_1, v_1\rangle &= |\beta|^2\langle v_2, v_2\rangle = \langle |\beta v_2, |\beta v_2\rangle \\ \langle u_1, v_1\rangle &= \langle \alpha u_2, \beta v_2\rangle\end{aligned}$$

it follows that, defining U on the subspace $M_1 + N_1$ by

$$U(au_1 + bv_1) = a\alpha u_2 + b\beta v_2$$

yields a unitary operator mapping $M_1 + N_1$ onto $M_2 + N_2$ with $UM_1 = M_2$ and $UN_1 = N_2$ because, for all a and b,

$$\begin{aligned}\|au_1 + bv_1\|^2 &= \langle au_1 + bv_1, au_1 + bv_1\rangle \\ &= |a|^2\langle u_1, u_1\rangle + 2\mathrm{Re}\, a\bar{b}\langle u_1, v_1\rangle + |b|^2\langle v_1, v_1\rangle \\ &= |a|^2\langle \alpha u_2, \alpha u_2\rangle + 2\mathrm{Re}\, a\bar{b}\langle \alpha u_2, \beta v_2\rangle + |b|^2\langle \beta v_2, \beta v_2\rangle \\ &= \|a\alpha u_2 + b\beta v_2\|^2\end{aligned}$$

Since $\dim(M_1 + N_1) = \dim(M_2 + N_2)$, we also have $\dim((M_1 + N_1)^\perp) = \dim((M_2 + N_2)^\perp)$. Choosing any unitary map of $(M_1 + N_1)^\perp$ onto $(M_2 + N_2)^\perp$, we can extend U to all of $\mathcal{H}$ so that it is unitary on $\mathcal{H}$ and satisfies $UM_1 = M_2$ and $UN_1 = N_2$. □

We are ready to show that the one parameter groups of adjoints of composition operators and analytic Toeplitz operators on H^2 are not unitarily equivalent.

Theorem 4. *There is no unitary operator U on H^2 such that $U^*C_{\varphi_t}U = T^*_{\psi_t}$ for every real number t.*

Proof. Suppose U is a unitary on H^2 such that $U^*C_{\varphi_t}U = T^*_{\psi_t}$ for every real number t. If f is a function in H^2 such that Uf is in the domain of H, then

$$\begin{aligned}\lim_{t\to 0}\frac{1}{t}(T^*_{\psi_t}f - f) &= \lim_{t\to 0}\frac{1}{t}(U^*C_{\varphi_t}Uf - U^*Uf) \\ &= U^*\lim_{t\to 0}\frac{1}{t}(C_{\varphi_t}Uf - Uf) \\ &= U^*H(Uf)\end{aligned}$$

This shows that f is in the domain of G^* and that $G^*f = U^*HUf$. That is, U^* takes the domain of H into the domain of G^*. Reversing the roles of G^* and H shows that U takes the domain of G^* into the domain of H. Thus, we see that U takes the domain of G^* onto the domain of H and that, therefore, H and G^* are unitarily equivalent.

This unitary equivalence implies that U takes eigenspaces of G^* onto eigenspaces of H. Since all the eigenspaces of G^* and H are one dimensional we can apply Lemma 3 to pairs of eigenspaces.

For example, if we take M_1 and M_2 to be the eigenspaces spanned by w_λ and v_λ, respectively, for $\lambda = 0$ and N_1 and N_2 to be the eigenspaces spanned by w_μ and v_μ, respectively, for $\mu = 1/4$.

Using Equation (2) to compute the square of the left hand side of the equality in the Lemma, we see that

$$\frac{|\langle w_\lambda, w_\mu\rangle|^2}{\|w_\lambda\|^2\|w_\mu\|^2} = \frac{(\frac{\cos(\pi/8)}{\cos(\pi/8)})^2}{1\frac{1}{\cos(\pi/4)}} = \frac{1}{\sqrt{2}}$$

On the other hand, from Equation (3), the right side is

$$\frac{|\langle v_\lambda, v_\mu\rangle|^2}{\|v_\lambda\|^2\|v_\mu\|^2} = \frac{(\frac{1\cos(\pi/8)}{\cos(\pi/8)})^2}{1\frac{\cos(\pi/8)^2}{\cos(\pi/4)}} = \frac{\cos(\pi/4)}{\cos(\pi/8)^2} = \frac{1}{\sqrt{2}\cos(\pi/8)^2}$$

Thus, the two sides are not equal, which contradicts the fact that U is unitary. □

The comparisons we are trying to make, if they will work at all, will work because w_λ and v_λ are eigenvectors corresponding to the eigenvalue λ for their respective operators. Equation (4), which expresses their relationship, is inconsistent with a unitary equivalence on the space H^2. Let us consider the possible restriction of our operators to another space.

Lemma 5. *If D is a bounded operator on the Hilbert space $\mathcal{H}$ and M is an invariant subspace for D, then $M^\perp$ is an invariant subspace for D^*. Furthermore, if r is an eigenvector for D with eigenvalue λ and $r = p + q$ where p is in M and q is in $M^\perp$, then either $q = 0$ or q is an eigenvector for the eigenvalue λ for the compression of D to $M^\perp$, which is the adjoint of the restriction of D^* to its invariant subspace $M^\perp$.*

Proof. Suppose x is in M and y is in $M^\perp$. Since M is invariant for D, then Dx is also in M. This means that $\langle Dx, y\rangle = 0$. However, $\langle x, D^*y\rangle = \langle Dx, y\rangle$, so we have, for all x in M and all y in $M^\perp$ that $\langle x, D^*y\rangle = 0$, so D^*y is also in $M^\perp$, which shows that $M^\perp$ is an invariant subspace for D^*.

We can write a block matrix for D with respect to the decomposition of $\mathcal{H}$ as $\mathcal{H} = M \oplus M^\perp$ so that

$$D = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where the 0 is in the lower left corner because M is invariant for D and A is the restriction of D to M. The operator C is the compression of D to $M^\perp$, that is, if P is the orthogonal projection of $\mathcal{H}$ onto $M^\perp$, then for y in $M^\perp$, $Cy = PDy$. Now $Dr = \lambda r$ and $r = p + q$ as in the hypothesis means that

$$\lambda r = Dr = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} Ap + Bq \\ Cq \end{pmatrix}$$

On the other hand, we have

$$\lambda r = \lambda\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \lambda p \\ \lambda q \end{pmatrix}$$

so that $Ap + Bq = \lambda p$ and $Cq = \lambda q$. The latter equation is exactly the penultimate statement of the Lemma.

Finally, the matrix for D^* is

$$D^* = \begin{pmatrix} A^* & 0 \\ B^* & C^* \end{pmatrix}$$

so the restriction of D^* to $M^\perp$, an invariant subspace, is C^* as the final statement of the Lemma asserts. □

Let [1] denote the subspace of H^2 spanned by the constant functions so that $H^2 = [1] \oplus zH^2$. Let P be the projection of H^2 onto the subspace zH^2, so that, in fact, for f in H^2, we have $Pf = f - f(0)$.

Corollary 6. *Let φ be a hyperbolic automorphism of the disk with fixed points ± 1 and let C_φ be the associated composition operator on H^2. Then* [1] *is an invariant subspace for C_φ and $[1]^\perp = zH^2$ is an invariant subspace for C_φ^*. Furthermore, if r is an eigenvector for C_φ with eigenvalue λ and $r = p + q$ where $p = \rho 1$ and $q \neq 0$ is in zH^2, then q is an eigenvector for the eigenvalue λ for the compression of C_φ to zH^2, which is the adjoint of the restriction of C_φ^* to its invariant subspace zH^2.*

Proof. This is a direct consequence of Lemma 5 with the observation that $C_\varphi 1 = 1 \circ \varphi = 1$ which means [1] is an invariant subspace for C_φ. □

Corollary 7. *Let ψ be the covering map of an annulus as above and let T_ψ be the associated composition operator on H^2. Then* [1] *is an invariant subspace for T_ψ^* and $[1]^\perp = zH^2$ is an invariant subspace for T_ψ. Furthermore, if r is an eigenvector for T_ψ^* with eigenvalue λ and $r = p + q$ where $p = \rho 1$ and $q \neq 0$ is in zH^2, then q is an eigenvector for the eigenvalue λ for the compression of T_ψ^* to zH^2, which is the adjoint of the restriction of T_ψ to its invariant subspace zH^2.*

Proof. This is a direct consequence of Lemma 5 with the observation that $T_\psi^* 1 = T_\psi^* K_0 = \overline{\psi(0)} K_0 = \overline{\psi(0)} 1$ which means [1] is an invariant subspace for T_ψ^*. □

For the composition operator group, we have been considering the eigenvectors w_λ; let $x_\lambda = P w_\lambda$. Because $w_\lambda(0) = 1$ for each λ under consideration, we have $w_\lambda = 1 + x_\lambda$ and this is the splitting of w_λ with respect to the decomposition $H^2 = [1] \oplus zH^2$ because 1 is in [1] and x_λ is in zH^2. This will be valuable because Lemma 5 says x_λ is an eigenvector for $e^{\lambda t}$ for the compressions of the operators C_{φ_t} to zH^2. Note that

$$\begin{aligned} \langle x_\lambda, x_\mu \rangle &= \langle w_\lambda - 1, w_\mu - 1 \rangle \\ &= \langle w_\lambda, w_\mu \rangle - \langle w_\lambda, 1 \rangle - \langle 1, w_\mu \rangle + \langle 1, 1 \rangle \\ &= \langle w_\lambda, w_\mu \rangle - 1 - 1 + 1 = \langle w_\lambda, w_\mu \rangle - 1 \end{aligned}$$

Similarly, for the adjoints of the Toeplitz operator group, we have we have been considering the eigenvectors v_λ; let $u_\lambda = P v_\lambda$. Because $v_\lambda(0) = 1$ for each λ under consideration, we have $v_\lambda = 1 + u_\lambda$ and this is the splitting of v_λ with respect to the decomposition $H^2 = [1] \oplus zH^2$ because 1 is in [1] and u_λ is in zH^2. This will be valuable because Lemma 5 says u_λ is an eigenvector for $e^{\lambda t}$ for the compressions of the operators $T_{\psi_t}^*$ to zH^2. Note that

$$\begin{aligned} \langle u_\lambda, u_\mu \rangle &= \langle v_\lambda - 1, v_\mu - 1 \rangle \\ &= \langle v_\lambda, v_\mu \rangle - \langle v_\lambda, 1 \rangle - \langle 1, v_\mu \rangle + \langle 1, 1 \rangle \\ &= \langle v_\lambda, v_\mu \rangle - 1 - 1 + 1 = \langle v_\lambda, v_\mu \rangle - 1 \end{aligned}$$

Putting these together with Equation (4), we have

$$\begin{aligned}2\langle u_\lambda, u_\mu\rangle &= 2(\langle v_\lambda, v_\mu\rangle - 1) = 2\langle v_\lambda, v_\mu\rangle - 2\\ &= (\langle w_\lambda, w_\mu\rangle + 1) - 2 = \langle w_\lambda, w_\mu\rangle - 1\\ &= \langle x_\lambda, x_\mu\rangle \end{aligned} \tag{5}$$

The following theorem summarizes our conclusions.

Theorem 8. *Let the vectors x_λ and u_λ be as described above for $-1/2 < \lambda < 1/2$. Then the following are true:*

(1) *The sets $\{x_\lambda : -1/2 < \lambda < 1/2\}$ and $\{u_\lambda : -1/2 < \lambda < 1/2\}$ are each linearly independent and have dense span in zH^2.*

(2) *If U is the operator obtained from*

$$U(x_\lambda) = \sqrt{2}u_\lambda$$

defining it to be linear from the span of $\{x_\lambda : -1/2 < \lambda < 1/2\}$ to the span of $\{u_\lambda : -1/2 < \lambda < 1/2\}$, then U is an isometry between these spans and can be further extended to a unitary operator of zH^2 onto itself.

(3) *The operator U gives a unitary equivalence of the one-parameter groups $\{C_{\varphi_t}^*|_{zH^2}\}_{t\in\mathbb{R}}$ and $\{T_{\psi_t}|_{zH^2}\}_{t\in\mathbb{R}}$. In particular, for each real number t,*

$$U\, C_{\varphi_t}^*\big|_{zH^2} = T_{\psi_t}\big|_{zH^2}\, U$$

Proof. (1) Corollary 2 shows that the projections of the w_λ and the v_λ, that is, the x_λ and the u_λ, are dense in zH^2. Notice that $w_0 = v_0 = 1$, so if some of the x_λ's or the u_λ's were linearly dependent, then the corresponding w_λ's with w_0 or the corresponding v_λ's together with v_0 would be dependent since each $w_\lambda = 1 + x_\lambda$ and each $v_\lambda = 1 + u_\lambda$. Since they are linearly independent, the conclusion follows.

(2) Equation (5) implies that the map U is isometric as a mapping from the span of the x_λ's to the span of the u_λ's. Indeed, if $-1/2 < \lambda_j < 1/2$ and a_j and b_j are complex numbers for $j = 1, 2, \cdots, n$, then

$$\begin{aligned}\langle\sum_{j=1}^n a_j x_{\lambda_j}, \sum_{k=1}^n b_k x_{\lambda_k}\rangle &= \sum_{j=1}^n\sum_{k=1}^n a_j\overline{b_k}\langle x_{\lambda_j}, x_{\lambda_k}\rangle = \sum_{j=1}^n\sum_{k=1}^n a_j\overline{b_k}2\langle u_{\lambda_j}, u_{\lambda_k}\rangle\\ &= \langle\sum_{j=1}^n a_j(\sqrt{2}u_{\lambda_j}), \sum_{k=1}^n b_k(\sqrt{2}u_{\lambda_k})\rangle\\ &= \langle U\left(\sum_{j=1}^n a_j x_{\lambda_j}\right), U\left(\sum_{k=1}^n b_k x_{\lambda_k}\right)\rangle\end{aligned}$$

Since these sets are dense in zH^2, the isometry U of the span of the x_λ's onto the span of the u_λ's can be extended to a unitary of zH^2 onto itself.

(3) For each $t \geq 0$, x_λ is an eigenvector for $e^{\lambda t}$ for the compressions of the operator C_{φ_t} to zH^2, so we see that

$$U(PC_{\varphi_t})x_\lambda = U(e^{\lambda t}x_\lambda) = e^{\lambda t}(\sqrt{2}u_\lambda)$$

On the other hand, for each $t \geq 0$, u_λ is an eigenvector for $e^{\lambda t}$ for the compressions of the operator $T^*_{\psi_t}$ to zH^2, so we see that

$$(PT^*_{\psi_t})Ux_\lambda = (PT^*_{\psi_t})(\sqrt{2}u_\lambda) = e^{\lambda t}(\sqrt{2}u_\lambda)$$

Since the span of the x_λ's and span of the u_λ's are both dense in zH^2, this means that $U(PC_{\varphi_t}) = (PT^*_{\psi_t})U$ on zH^2. Taking adjoints, we see this is equivalent to $(PC_{\varphi_t})^*U^* = U^*(PT^*_{\psi_t})^*$. Since the adjoints of the compressions are the restrictions of the adjoints, we get $C_{\varphi_t}^*\big|_{zH^2} U^* = U^* \, T_{\psi_t}\big|_{zH^2}$ which is equivalent to the result of (3). Since the same unitary works for every t, the groups are unitarily equivalent. □

Corollary 9. *For each real number t, the operators $C_{\varphi_t}^*\big|_{zH^2}$ and T_{ψ_t} on H^2 are unitarily equivalent.*

Proof. The operator T_z is a unitary map of H^2 onto zH^2. For any function h in $H^\infty(\mathbb{D})$, the analytic Toeplitz operator T_h on H^2 is unitarily equivalent to $T_h|_{zH^2}$ because

$$T_z^* \, T_h|_{zH^2} \, T_z = T_z^* T_h T_z = T_z^* T_z T_h = T_h$$

Thus, $T_{\psi_t}|_{zH^2}$ is unitarily equivalent to T_{ψ_t} on H^2 and the corollary follows. □

Theorem 8 gives us a new, easy proof of the subnormality of the operators $C_{\varphi_t}^*\big|_{zH^2}$. The earlier proof of the stronger result that $C_{\varphi_t}^*$ is subnormal on all of H^2 depends on the ideas in the papers [12, 7, 14, 5] noted earlier. The result of Corollary 10 follows immediately from that because the restriction of a subnormal operator to an invariant subspace is also subnormal.

Corollary 10. *For each real number t, the operators $C_{\varphi_t}^*\big|_{zH^2}$ are subnormal.*

Proof. Each T_{ψ_t} is subnormal on H^2. □

There are clearly connections between analytic Toeplitz operators and composition operators, for example (see [1, 2]), the commutant of an analytic Toeplitz operator often consists of the algebra generated by the composition operators that commute with it and the analytic Toeplitz operators. On the other hand, it is often thought that the structures of these operators are quite different from each other. The results of this paper show that this is not always the case and suggest that this issue needs further examination.

References

[1] C. C. Cowen, The commutant of an analytic Toeplitz operator, *Trans. Amer. Math. Soc.* **239**(1978), 1–31.

[2] C. C. Cowen, The commutant of an analytic Toeplitz operator, II, *Indiana Univ. Math. J.* **29**(1980), 1–12.

[3] C. C. Cowen, Subnormality of the Cesaro operator and a semigroup of composition operators, *Indiana Univ. Math. J.* **33**(1984), 305–318.

[4] C. C. Cowen, Linear fractional composition operators on H^2, *J. Integral Equations Operator Theory* **11**(1988), 151–160.

[5] C. C. Cowen and T. L. Kriete, Subnormality and composition operators on H^2, *J. Functional Analysis* **81**(1988), 298–319.

[6] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, 1995.
[7] J. A. Deddens, Analytic Toeplitz and composition operators, *Canadian J. Math.* **24**(1972), 859–865.
[8] R. G. Douglas. "Banach Algebra Techniques in Operator Theory," Academic Press, New York, 1972.
[9] A. Erdelyi, et. al., "Higher Transcendental Functions, Vol. 1," McGraw-Hill, New York, 1953.
[10] A. Erdelyi, et. al., "Tables of Integral Transforms, Vol. 1," McGraw-Hill, New York, 1954.
[11] T. L. Kriete and D. Trutt, The Cesàro operator in ℓ^2 is subnormal, *Amer. J. Math.* **93**(1971), 215–225.
[12] T. L. Kriete and D. Trutt, On the Cesàro operator, *Indiana Univ. Math. J.* **24**(1974), 197–214.
[13] Maple, version 12.02, 2008, Waterloo Maple, Inc., Waterloo.
[14] E. A. Nordgren, P. Rosenthal, and F. S. Wintrobe, Invertible composition operators on H^p, *J. Functional Analysis* **73**(1987), 324–344.
[15] H. Radjavi and P. Rosenthal,"Invariant subspaces," Springer-Verlag, New York, 1973.
[16] S. M. Selby, editor, "CRC Standard Mathematical Tables", 16^{th} edition, 1968, Chemical Rubber Company, Cleveland.

Department of Mathematical Sciences, Indiana University-Purdue University Indianapolis, Indianapolis, IN 46202, USA
E-mail address: ccowen@iupui.edu

Departamento de Matemáticas, Universidad de Zaragoza e IUMA, Plaza San Francisco s/n, 50009 Zaragoza, SPAIN.
E-mail address: eva@unizar.es