HERMITIAN WEIGHTED COMPOSITION OPERATORS AND BERGMAN EXTREMAL FUNCTIONS

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Abstract. Weighted composition operators have been related to products of composition operators and their adjoints and to isometries of Hardy spaces. In this paper, Hermitian weighted composition operators on weighted Hardy spaces of the unit disk are studied. In particular, necessary conditions are provided for a weighted composition operator to be Hermitian on such spaces. On weighted Hardy spaces for which the kernel functions are $(1-\overline{w}z)^{-\kappa}$ for $\kappa \geq 1$, including the standard weight Bergman spaces, the Hermitian weighted composition operators are explicitly identified and their spectra and spectral decompositions are described. Some of these Hermitian operators are part of a family of closely related normal weighted composition operators. In addition, as a consequence of the properties of weighted composition operators, we compute the extremal functions for the subspaces associated with the usual atomic inner functions for these weighted Bergman spaces and we also get explicit formulas for the projections of the kernel functions on these subspaces.

1. Introduction

In this paper, we give necessary conditions (Theorem 3) on the symbols f and φ for $W_{f,\varphi}$ to be a Hermitian weighted composition operator on a weighted Hardy space. For the standard weight Bergman spaces A^2_α for $\alpha \geq 0$, all of which are also weighted Hardy spaces, we establish the converse (Theorem 6) and identify all of the Hermitian weighted composition operators for these spaces. In Sections 2 and 3, we identify the spectra and spectral decompositions of these operators. Section 3 covers the Hermitian weighted composition operators with continuous spectra and we establish a unitary equivalence (Theorem 17) between these operators and multiplication operators on an $L^2([0,1],\mu)$ space. In addition, we show that these operators are part of an analytic semigroup of normal weighted composition operators (Corollary 19). In the final section of the paper, we apply the results of Section 3 to find the extremal functions (Theorem 25) for the invariant subspaces for multiplication by z in the Bergman spaces A^2_α for $\alpha \geq 0$ that are

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associated with the usual atomic inner functions on the disk. Finally, explicit formulas for the kernel functions for these subspaces are computed (Theorem 26).

Weighted composition operators have been studied occasionally over the past few decades, but have usually arisen in answering other questions related to operators on spaces of analytic functions, such as questions about multiplication operators or composition operators. For example, Forelli [13] showed that the only isometries of H^p for $p \neq 2$ are weighted composition operators and that the isometries for H^p with $p \neq 2$ have analogues that are isometries of H^2 (but there are also many other isometries of H^2). Weighted composition operators also arise in the description of commutants of analytic Toeplitz operators (see for example [3, 4]) and in the adjoints of composition operators (see for example [6, 7, 8, 10]). Only recently have investigators begun to study the properties of weighted composition operators in general (for example, see [15]). Our goal in this paper is to characterize the self-adjoint weighted composition operators on a very broad class of spaces, the weighted Hardy spaces.

A Hilbert space $\mathcal{H}$ whose vectors are functions analytic on the unit disk $\mathbb{D}$ is called [9, p. 14] a *weighted Hardy space* if the polynomials are dense and the monomials 1, z, z^2, $\cdots$, are an orthogonal set of non-zero vectors in $\mathcal{H}$. Each weighted Hardy space is characterized by a weight sequence, β, defined for each non-negative integer j by $\beta(j) = \|z^j\|$. For this paper, we will assume that the norm has been scaled so that $\beta(0) = \|1\| = 1$. For a given weight sequence, β, the corresponding weighted Hardy space will be denoted $H^2(\beta)$, and its inner product is given by

$$\langle \sum_{j=0}^{\infty} a_j z^j, \sum_{j=0}^{\infty} c_j z^j \rangle = \sum_{j=0}^{\infty} a_j \overline{c_j} \beta(j)^2$$

for functions in $H^2(\beta)$.

If f is analytic on the open unit disk, $\mathbb{D}$, and φ is an analytic map of the unit disk into itself, the *weighted composition operator on $H^2(\beta)$ with symbols f and φ* is the operator $(W_{f,\varphi}h)(z) = f(z)h(\varphi(z))$ for h in $H^2(\beta)$. Letting T_f denote the analytic multiplication operator given by $T_f(h) = fh$ and C_φ the composition operator given by $C_\varphi(h) = h \circ \varphi$ for h in $H^2(\beta)$, if T_f and C_φ are both bounded operators, then clearly $W_{f,\varphi}$ is bounded on $H^2(\beta)$ and

$$\|W_{f,\varphi}\| = \|T_f C_\varphi\| \leq \|T_f\| \|C_\varphi\|$$

Although it will have little impact on our work, for any weighted Hardy space, if T_f is bounded, then f is in H^∞, but it is not necessary for T_f to be bounded for $W_{f,\varphi}$ to be bounded (see [15]).

In an earlier paper, Cowen and Ko [11] identified the Hermitian weighted composition operators on the standard Hardy Hilbert space, H^2, that is, in the case $\beta(j) \equiv 1$. Specifically, they showed that if $W_{f,\varphi}$ is Hermitian, then f and φ are related linear fractional maps that can be separated into three distinct cases, some for which $W_{f,\varphi}$ is a real multiple of a unitary operator as well as being Hermitian, some for which $W_{f,\varphi}$ is compact, and some for which $W_{f,\varphi}$ has no eigenvalues. In each case, the spectral measures were computed and in some of the cases, further information was provided. Generally speaking, the results for more general spaces $H^2(\beta)$ divide into the same three cases. Many of the techniques used in that paper carry over to cases studied in this paper, as will be noted in the specific results. In the cases in which the proof in the general weighted Hardy space is essentially the same as in the usual Hardy space, we will omit the proof and refer the reader to [11].

In this paper, we give necessary conditions (Theorem 3) on the symbols f and φ for $W_{f,\varphi}$ to be a Hermitian weighted composition operator on $H^2(\beta)$. We prove the converse of this theorem (Theorem 6) for weighted Hardy spaces where the kernels for evaluation of the functions in the space are $K_w(z) = (1 - \overline{w}z)^{-\kappa}$ for $\kappa \geq 1$; we will write $H^2(\beta_\kappa)$ for this weighted Hardy space. In the classical Hardy space, $\kappa = 1$ and in the Bergman space, $\kappa = 2$. Spaces with other values of κ are studied in [21], for example, and in [9, p. 27], it is noted that these spaces are equivalent to the "standard weight Bergman spaces". In particular, for $\kappa > 1$, the weighted Bergman space $A^2_{\kappa-2}$ consists of the same functions as $H^2(\beta_\kappa)$ and

$$\int_{\mathbb{D}} |f(z)|^2 (\kappa - 1)(1 - |z|^2)^{\kappa-2} \, \frac{dA}{\pi}$$

is another expression for the norm for the space.

Specifically, we show that, as in the usual Hardy space, the Hermitian weighted composition operators fall into three classes, the compact weighted composition operators, the multiples of isometries, (both covered in Section 2), and those that have no eigenvalues (Section 3). In the first two cases, we identify the eigenvalues and eigenvectors in $H^2(\beta_\kappa)$, (providing spectral resolutions for these operators) and in the third case, for general spaces $H^2(\beta_\kappa)$, we see that the Hermitian weighted composition operators are cyclic, are part of an analytic semigroup that includes normal weighted composition operators, and have no eigenvalues. We find measures $\mu = \mu_\kappa$ on $[0, 1]$ that allow us to identify the spectral resolutions and a unitary equivalence with multiplication operators $\{M_{x^t}\}$ on $L^2(\mu)$. In addition, as a consequence of the calculation of the spectral measures, using properties of the weighted composition operators and the unitary equivalence, we compute the extremal functions associated with the Bergman shift-invariant subspaces associated with the usual atomic inner functions and we find the reproducing kernel functions for these subspaces (Theorems 25 and 26).

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We begin by proving a result that allows us to choose standard forms for the operators under study.

Proposition 1. *Let f be analytic on the unit disk and let φ be an analytic map of the unit disk into itself. For θ a real number, let U_θ be the composition operator given by $(U_\theta h)(z) = h(e^{i\theta} z)$ for h in $H^2(\beta)$. The operator U_θ is unitary on $H^2(\beta)$, and if $W_{f,\varphi}$ is bounded, then*

$$U_\theta^* W_{f,\varphi} U_\theta = W_{\tilde{f},\widetilde{\varphi}}$$

where $\tilde{f}(z) = f(e^{-i\theta} z)$ and $\widetilde{\varphi}(z) = e^{i\theta}\varphi(e^{-i\theta} z)$

Proof. The calculation is essentially the same as in the proof of the corresponding result in [11], and is omitted. □

This proposition will permit us to choose convenient symbols for the weighted composition operators we study without losing any generality of the properties of the operators we are trying to understand.

Corollary 2. *If f analytic in the unit disk and φ an analytic map of the disk into itself determine a bounded weighted composition operator, $W_{f,\varphi}$, there are g analytic in the disk and ψ an analytic map of the unit disk into itself with $\psi(0) \geq 0$ so that the weighted composition operator $W_{g,\psi}$ is unitarily equivalent to $W_{f,\varphi}$.*

Proof. Choose θ in Proposition 1 so that $\widetilde{\varphi}(0) = e^{i\theta}\varphi(e^{-i\theta}0) = e^{i\theta}\varphi(0)$ is non-negative. Letting $g = \tilde{f}$ and $\psi = \widetilde{\varphi}$ satisfies the conclusion of the Corollary. □

As in Proposition 1.3 of [11], if f and φ determine a bounded weighted composition operator on $H^2(\beta)$ and if g and h are functions in $H^2(\beta)$ such that gh is also in $H^2(\beta)$, then $(W_{f,\varphi})(gh) = h \circ \varphi W_{f,\varphi} g = g \circ \varphi W_{f,\varphi} h$.

Following the book of Cowen and MacCluer [9, p. 16], the *generating function* for the space $H^2(\beta)$ is the function

$$k(z) = \sum_{j=0}^{\infty} \frac{z^j}{\beta(j)^2}$$

The function $k(z)$ is analytic on the unit disk and, for each w in the disk, the function $K_w(z) = k(\overline{w}z)$ belongs to $H^2(\beta)$. The K_w are the kernel functions for $H^2(\beta)$, that is, the functions for which $f(w) = \langle f, K_w \rangle$ and this implies $\|K_w\|^2 = k(|w|^2)$.

First, we wish to characterize the functions f and φ of bounded Hermitian weighted composition operators.

Theorem 3. *Let k be the generating function for $H^2(\beta)$. If $W_{f,\varphi}$ is a bounded Hermitian weighted composition operator on $H^2(\beta)$, then $f(0)$ and $\varphi'(0)$ are real,*

$$f(z) = ck(\overline{a_0}z) = cK_{a_0}(z) \qquad \text{and} \qquad \varphi(z) = a_0 + a_1\beta(1)^2 z \frac{k'(\overline{a_0}z)}{k(\overline{a_0}z)}$$

where $a_0 = \varphi(0)$, $a_1 = \varphi'(0)$, and $c = f(0)$.

Proof. Let $W_{f,\varphi}$ be a bounded Hermitian weighted composition operator on $H^2(\beta)$. Let w and z be points of the open unit disk, $\mathbb{D}$. Now, from the definition of $W_{f,\varphi}$,

$$(W_{f,\varphi}K_w)(z) = f(z)K_w(\varphi(z)) = f(z)k(\overline{w}\varphi(z))$$

and, using the special properties of $W^*_{f,\varphi}$ acting on kernel functions,

$$(W^*_{f,\varphi}K_w)(z) = \overline{f(w)}K_{\varphi(w)}(z) = \overline{f(w)}k(\overline{\varphi(w)}z)$$

Because $W_{f,\varphi}$ is self-adjoint, we have

$$f(z)k(\overline{w}\varphi(z)) = \overline{f(w)}k(\overline{\varphi(w)}z) \tag{1}$$

Putting $w = 0$, this means

$$f(z)k(\overline{0}\varphi(z)) = \overline{f(0)}k(\overline{\varphi(0)}z)$$

Recalling that we have assumed that $k(0) = 1/\beta(0)^2 = 1$ and that $a_0 = \varphi(0)$, we have

$$f(z) = \overline{f(0)}k(\overline{a_0}z)$$

Putting $z = 0$ also, we get

$$f(0) = \overline{f(0)} k(\overline{a_0} 0) = \overline{f(0)}$$

so $c = f(0)$ is real, and we have

$$f(z) = ck(\overline{a_0} z) = cK_{a_0}(z)$$

Since we are not interested in the case $f \equiv 0$ which gives $W_{f,\varphi} = 0$, we will assume c is a non-zero real number.

Using this expression for f in Equation (1), we have

$$ck(\overline{a_0} z)k(\overline{w}\varphi(z)) = \overline{ck(\overline{a_0} w)k(\overline{\varphi(w)} z)} = c\overline{k(\overline{a_0} w)}k(\overline{\overline{\varphi(w)} z})$$

The Taylor coefficients of k are real numbers, so $\overline{k(u)} = k(\overline{u})$ for any complex number u. The above equality can be rewritten as

$$k(\overline{a_0} z)k(\overline{w}\varphi(z)) = k(a_0 \overline{w})k(\overline{\varphi(w)} z)$$

Taking the derivative with respect to z gives

$$k'(\overline{a_0} z)\overline{a_0} k(\overline{w}\varphi(z)) + k(\overline{a_0} z)k'(\overline{w}\varphi(z))\overline{w}\varphi'(z) = k(a_0 \overline{w})k'(\overline{\varphi(w)} z)\overline{\varphi(w)}$$

Letting $z = 0$, we have

$$k'(0)\overline{a_0} k(\overline{w}\varphi(0)) + k(0)k'(\overline{w}\varphi(0))\overline{w}\varphi'(0) = k(a_0 \overline{w})k'(0)\overline{\varphi(w)}$$

or, noting that $k'(0) = 1/\beta(1)^2$ and $\varphi'(0) = a_1$, taking conjugates and solving for φ, we have

$$\varphi(w) = a_0 + \overline{a_1}\beta(1)^2 w \frac{k'(\overline{a_0} w)}{k(\overline{a_0} w)}$$

Finally, taking the derivative again, and setting $w = 0$, we get

$$\varphi'(0) = \overline{a_1}\beta(1)^2 \frac{k'(\overline{a_0} 0)}{k(\overline{a_0} 0)} = \beta(1)^2 \overline{a_1} \frac{k'(0)}{k(0)} = \overline{a_1}$$

Since $\varphi'(0) = a_1$, this latter equality says $\varphi'(0) = \overline{\varphi'(0)}$ which means that $\varphi'(0)$ is real and the proof is complete. □

We prove the converse of the above theorem for weighted Hardy spaces where the point evaluation kernel is given by $K_w(z) = (1 - \overline{w}z)^{-\kappa}$ for $\kappa \geq 1$, that is, the generating function is $k(z) = (1 - z)^{-\kappa}$. For $\kappa > 1$, since the weighted Bergman space $A^2_{\kappa-2}$ consists of the same functions as $H^2(\beta_\kappa)$ and

$$\int_{\mathbb{D}} |f(z)|^2 (\kappa - 1)(1 - |z|^2)^{\kappa-2} \, \frac{dA}{\pi}$$

gives the norm for the space, it is clear that multiplication by an H^∞ function gives rise to a bounded operator on $H^2(\beta_\kappa)$. It also follows that if f is in $H^2(\beta_\kappa)$ and g is analytic on $\mathbb{D}$ with $|g(z)| \leq |f(z)|$ for all z in $\mathbb{D}$, then g is also in $H^2(\beta_\kappa)$. These facts and similar ones will be used below without special reference.

Corollary 4. *For $\kappa \geq 1$, let $H^2(\beta_\kappa)$ be the weighted Hardy space with kernel function $K_w(z) = (1 - \overline{w}z)^{-\kappa}$. If $W_{f,\varphi}$ is a bounded Hermitian weighted composition operator on $H^2(\beta_\kappa)$, then*

$$f(z) = c(1 - \overline{a_0} z)^{-\kappa} = cK_{\varphi(0)}(z) \quad \text{and} \quad \varphi(z) = a_0 + \frac{a_1 z}{1 - \overline{a_0} z}$$

where $a_0 = \varphi(0)$ is a complex number and $a_1 = \varphi'(0)$ is real number such that φ maps the unit disk into itself and $c = f(0)$ is a non-zero real number.

Proof. The generating function for $H^2(\beta_\kappa)$ is $k(z) = (1-z)^{-\kappa}$, so $k(0) = 1$, as was assumed for the scaling. We also see that $k'(z) = \kappa(1-z)^{-\kappa-1}$ so $1/\beta(1)^2 = k'(0) = \kappa$ and $\beta(1)^2 = 1/\kappa$.

From this, we conclude that

$$\varphi(z) = a_0 + a_1\beta(1)^2 z\frac{k'(\overline{a_0}z)}{k(\overline{a_0}z)} = a_0 + a_1\frac{1}{\kappa}z\frac{\kappa(1-\overline{a_0}z)^{-\kappa-1}}{(1-\overline{a_0}z)^{-\kappa}} = a_0 + \frac{a_1 z}{1-\overline{a_0}z}$$

□

Not surprisingly, this duplicates the corresponding implication of Theorem 2.1 of [11] for the standard Hardy space for which $\kappa = 1$. Since the function φ is independent of κ, Corollary 2.3 of [11] describes the conditions on a_0 and a_1 that result in the map φ taking the unit disk into itself. This result is recorded below.

Lemma 5. *Let a_1 be real. Then $\varphi(z) = a_0 + a_1 z/(1-\overline{a_0}z)$ maps the open unit disk into itself if and only if*

$$|a_0| < 1 \quad \text{and} \quad -1 + |a_0|^2 \le a_1 \le (1-|a_0|)^2 \tag{2}$$

We are now ready to state the necessary and sufficient conditions for the functions f and φ that guarantee $W_{f,\varphi}$ is a bounded Hermitian weighted composition operator on $H^2(\beta_\kappa)$.

Theorem 6. *For $\kappa \ge 1$, let $H^2(\beta_\kappa)$ be the weighted Hardy space with kernel function $K_w(z) = (1-\overline{w}z)^{-\kappa}$. If the weighted composition operator $W_{f,\varphi}$ is bounded and Hermitian on $H^2(\beta_\kappa)$, then $f(0)$ and $\varphi'(0)$ are real and*

$$\varphi(z) = a_0 + a_1 z/(1-\overline{a_0}z) \quad \text{and} \quad f(z) = c(1-\overline{a_0}z)^{-\kappa} = cK_{\varphi(0)}(z)$$

where $a_0 = \varphi(0)$, $a_1 = \varphi'(0)$, and $c = f(0)$.

Conversely, suppose a_0 is in $\mathbb{D}$, and suppose c and a_1 are real numbers. If $\varphi(z) = a_0 + a_1 z/(1-\overline{a_0}z)$ maps the unit disk into itself and $f(z) = c(1-\overline{a_0}z)^{-\kappa}$, then the weighted composition operator $W_{f,\varphi}$ is Hermitian and bounded on $H^2(\beta_\kappa)$.

Proof. The first statement follows from Corollary 4. If a_0, a_1, c, f, and φ are as in the hypothesis, the results of [9, p. 27 and Section 3.1] show that $W_{f,\varphi}$ is a bounded operator on $H^2(\beta_\kappa)$. Also, a straightforward calculation shows that Equation (1) holds for all z and w in the disk, which means that $W_{f,\varphi}$ is Hermitian. □

When $a_0 = 0$, the resulting operator is a constant multiple of a composition operator. In this case, $W_{f,\varphi}$ self adjoint implies $\varphi(z) = rz$ with $-1 \le r \le 1$. For $r = \pm 1$, the composition operators are $C_\varphi = \pm I$ and for $-1 < r < 1$, the composition operators are easily understood compact operators, so we will only consider non-zero values for a_0. Furthermore, without loss of generality, we may take a_0 to be a positive number by using the transformation in Corollary 2. With these hypotheses added, in the remainder of the paper, we will look at the different cases suggested by Lemma 5: $-1 + a_0^2 < a_1 < (1-a_0)^2$, $a_1 = -1 + a_0^2$, and $a_1 = (1-a_0)^2$.

2. Compact and isometric Hermitian weighted composition operators

We will see that the Hermitian weighted composition operators are all compact when the parameters defining φ satisfy $-1+|a_0|^2 < a_1 < (1-|a_0|)^2$ and they are isometries when $a_1 = -1+|a_0|^2$. Using Proposition 1, without loss of generality, we may assume $0 < a_0 < 1$.

Easy calculations (see, for example, the discussion following Corollary 2.3 in [11]), show that when $0 < a_0 < 1$ and $-1+a_0^2 < a_1 < (1-a_0)^2$ the function φ maps the closed unit disk into the open unit disk. In particular, since φ is continuous and maps the interval $[-1,1]$ into $(-1,1)$, it must have a fixed point in $(-1,1)$. Similarly, when $a_1 = -1+a_0^2$, we see that

$$\varphi(z) = a_0 + \frac{a_1 z}{1-a_0 z} = a_0 + \frac{(-1+a_0^2)z}{1-a_0 z} = \frac{a_0 - z}{1-a_0 z}$$

so that φ is an automorphism of the unit disk that satisfies $\varphi\circ\varphi(z) = z$. Moreover, since $\varphi(1) = -1$ and φ maps $[-1,1]$ onto itself, it must have a fixed point in $(-1,1)$ in this case as well. The most important part of the argument in these cases depends on the presence of the fixed point.

First, we show that in the first case, the operators are Hilbert-Schmidt and therefore they are compact operators.

Theorem 7. *For $\kappa \geq 1$, let $H^2(\beta_\kappa)$ be the weighted Hardy space with kernel function $K_w(z) = (1-\overline{w}z)^{-\kappa}$. Suppose*

$$\varphi(z) = a_0 + a_1 z/(1-\overline{a_0}z) \quad \textit{and} \quad f(z) = c(1-\overline{a_0}z)^{-\kappa} = cK_{\varphi(0)}(z)$$

where c is a real number. If $0 < a_0 < 1$ and $-1+a_0^2 < a_1 < (1-a_0)^2$ then $W_{f,\varphi}$ is Hilbert-Schmidt.

Proof. Let $f_n(z) = \beta(n)^{-1}z^n$. Then $\{f_n\}_{n=0}^\infty$ is an orthonormal basis of the weighted Hardy space $H^2(\beta_\kappa)$. We will show that $\sum_{n=0}^\infty \|W_{f,\varphi}(f_n)\|^2$ is finite which will show $W_{f,\varphi}$ is Hilbert-Schmidt [9, Theorem 3.23]. The comments above imply that in this case, there is $r < 1$ so that $|\varphi(z)| < r$ for z in $\mathbb{D}$. Now, because the spaces $H^2(\beta_\kappa)$ and $A^2_{\kappa-2}(\mathbb{D})$ are equivalent, there are constants M and M' so that

$$\begin{aligned}
\sum_{j=0}^\infty \|W_{f,\varphi}(f_j)\|^2 &\leq M \sum_{j=0}^\infty \int_{\mathbb{D}} c^2 |K_{a_0}(z)|^2 \beta(j)^{-2} |\varphi(z)|^{2j} (1-|z|)^{\kappa-2}\,\frac{dA}{\pi} \\
&\leq M \frac{c^2}{(1-a_0)^{2\kappa}} \sum_{j=0}^\infty \int_{\mathbb{D}} \beta(j)^{-2} r^{2j} (1-|z|)^{\kappa-2}\,\frac{dA}{\pi} \\
&= \frac{c^2}{(1-a_0)^{2\kappa}} M' \sum_{j=0}^\infty \beta(j)^{-2} r^{2j} \\
&= \frac{c^2}{(1-a_0)^{2\kappa}} M' k(r^2)
\end{aligned}$$

Thus $\sum_{n=0}^\infty \|W_{f,\varphi}(f_n)\|^2$ is finite. □

We will next explore the eigenvectors of these operators.

Lemma 8. *Suppose $a_1 \neq 0$ and a_0 are real numbers such that $0 < a_0 < 1$ and $\varphi(z) = a_0 + \dfrac{a_1 z}{1 - a_0 z}$ maps the disk into itself with fixed point α in the disk. This implies*

$$1 - \alpha\varphi(z) = \frac{(1 - a_0\alpha)(1 - \alpha z)}{1 - a_0 z} \tag{3}$$

$$\varphi'(\alpha) = \frac{(a_1 - a_0^2) + a_0\alpha}{1 - a_0\alpha} \tag{4}$$

and

$$\text{for} \quad g(z) = \frac{\alpha - z}{1 - \alpha z} \qquad \text{then} \quad g(\varphi(z)) = \varphi'(\alpha) g(z) \tag{5}$$

Proof. That α is a fixed point of φ means $\alpha = a_0 + \dfrac{a_1\alpha}{1 - a_0\alpha} = \dfrac{a_1\alpha + a_0 - a_0^2\alpha}{1 - a_0\alpha}$ so

$$\alpha(1 - a_0\alpha) \quad = \quad \alpha - a_0\alpha^2 = a_1\alpha + a_0 - a_0^2\alpha \tag{6}$$

and, rearranging the terms in this equation, we get

$$\alpha - a_0 \quad = \quad a_1\alpha + a_0\alpha^2 - a_0^2\alpha = \alpha(a_1 + a_0\alpha - a_0^2) \tag{7}$$

If $a_1 + a_0\alpha - a_0^2 = 0$ then $\alpha = a_0$ and $a_1 = 0$, contrary to our hypothesis, so we may also assume $a_1 + a_0\alpha - a_0^2 \neq 0$. Similarly, if α were 0, we would have $a_0 = 0$, so $\alpha \neq 0$.

The definition of φ means $\varphi(z) - a_0 = a_1 z/(1 - a_0 z)$, so for $z = \alpha$, we get

$$\alpha - a_0 = \varphi(\alpha) - a_0 = \frac{a_1\alpha}{1 - a_0\alpha}$$

This, together with Equation (7), gives

$$\frac{a_1}{1 - a_0\alpha} = \frac{\alpha - a_0}{\alpha} = a_1 + a_0\alpha - a_0^2$$

Taking the derivative of φ, we get $\varphi'(\alpha) = a_1/(1 - a_0\alpha)^2$ and we see

$$\varphi'(\alpha) = \frac{a_1}{(1 - a_0\alpha)^2} = \left(\frac{a_1}{1 - a_0\alpha}\right)\frac{1}{1 - a_0\alpha} = \frac{a_1 + a_0\alpha - a_0^2}{1 - a_0\alpha}$$

as we wanted to prove.

Writing φ as $\varphi(z) = \left(a_0 + (a_1 - a_0^2)z\right)/(1 - a_0 z)$, we get

$$\begin{aligned} 1 - \alpha\varphi(z) &= 1 - \alpha\left(\frac{a_0 + (a_1 - a_0^2)z}{1 - a_0 z}\right) \\ &= \frac{(1 - a_0\alpha) - (a_1\alpha + a_0 - a_0^2\alpha)z}{1 - a_0 z} \end{aligned}$$

Using Equation (6), this becomes

$$1 - \alpha\varphi(z) = \frac{(1 - a_0\alpha) - \alpha(1 - a_0\alpha)z}{1 - a_0 z} = \frac{(1 - a_0\alpha)(1 - \alpha z)}{1 - a_0 z} \tag{8}$$

as we wished to show.

From the definition of g and Equation (8), we see

$$\begin{aligned} g(\varphi(z)) &= \frac{\alpha-\varphi(z)}{1-\alpha\varphi(z)} = \frac{(\alpha-\varphi(z))(1-a_0z)}{(1-a_0\alpha)(1-\alpha z)} = \frac{(\alpha-a_0-\frac{a_1z}{1-a_0z})(1-a_0z)}{(1-a_0\alpha)(1-\alpha z)} \\ &= \frac{\alpha-a_0-\alpha a_0z+a_0^2z-a_1z}{(1-a_0\alpha)(1-\alpha z)} = \frac{(\alpha-a_0)-z(a_1+a_0\alpha-a_0^2)}{(1-a_0\alpha)(1-\alpha z)} \end{aligned}$$

Using Equation (7) and our calculation of $\varphi'(\alpha)$ in this result, we get

$$\begin{aligned} g(\varphi(z)) &= \frac{\alpha(a_1+a_0\alpha-a_0^2)-z(a_1+a_0\alpha-a_0^2)}{(1-a_0\alpha)(1-\alpha z)} \\ &= \left(\frac{a_1+a_0\alpha-a_0^2}{1-a_0\alpha}\right)\left(\frac{\alpha-z}{1-\alpha z}\right) = \varphi'(\alpha)g(z) \end{aligned}$$

which completes the proof of the lemma. □

We are now ready to collect the information we have to characterize the eigenvectors and eigenvalues of these weighted composition operators.

Theorem 9. *For $\kappa \geq 1$, let $H^2(\beta_\kappa)$ be the weighted Hardy space with kernel function $K_w(z) = (1-\overline{w}z)^{-\kappa}$. Suppose a_0 and a_1 are real numbers such that $0 < a_0 < 1$ and $\varphi(z) = a_0 + \dfrac{a_1z}{1-a_0z}$ maps the disk into itself with fixed point α in the disk and $f(z) = c(1-a_0z)^{-\kappa} = cK_{\varphi(0)}(z)$ for some real number c. For each non-negative integer j, the function*

$$g_j(z) = \frac{1}{(1-\alpha z)^\kappa}\left(\frac{\alpha-z}{1-\alpha z}\right)^j$$

is an eigenvector of the operator $W_{f,\varphi}$ with eigenvalue $f(\alpha)\varphi'(\alpha)^j$.

Proof. Since g_j is a bounded analytic map on the unit disk it belongs to $H^2(\beta_\kappa)$.

$$W_{f,\varphi}(g_j)(z) = \frac{c}{(1-a_0z)^\kappa}\frac{1}{(1-\alpha\varphi(z))^\kappa}\left(\frac{\alpha-\varphi(z)}{1-\alpha\varphi(z)}\right)^j$$

Now using Lemma 8, we get,

$$\begin{aligned} W_{f,\varphi}(g_j)(z) &= \frac{c}{(1-a_0z)^\kappa}\frac{1}{(1-a_0\alpha)^\kappa}\frac{(1-a_0z)^\kappa}{(1-\alpha z)^\kappa}\left(\varphi'(\alpha)\left(\frac{z-\alpha}{\alpha z-1}\right)\right)^j \\ &= \frac{c}{(1-a_0\alpha)^\kappa}\left(\varphi'(\alpha)\right)^j\frac{1}{(1-\alpha z)^\kappa}\left(\frac{\alpha-z}{1-\alpha z}\right)^j = f(\alpha)\varphi'(\alpha)^j g_j(z) \end{aligned}$$

□

We can apply this to the case in which $W_{f,\varphi}$ is compact.

Corollary 10. *For $\kappa \geq 1$, let $H^2(\beta_\kappa)$ be the weighted Hardy space with kernel function $K_w(z) = (1-\overline{w}z)^{-\kappa}$. Suppose a_0, a_1, and c are real numbers such that $0 < a_0 < 1$ and $-1+a_0^2 < a_1 < (1-a_0)^2$. If φ, f, and g_j are defined as in Theorem 9 and α is the fixed point of φ in the disk, then $\{g_j\}_{j=0}^\infty$ is an orthogonal basis for $H^2(\beta_\kappa)$ consisting of eigenvectors for $W_{f,\varphi}$ with $W_{f,\varphi}(g_j) = f(\alpha)\varphi'(\alpha)^j g_j$. In particular, $W_{f,\varphi}$ is a compact Hermitian operator and the spectrum of $W_{f,\varphi}$ is $\{0\} \cup \{f(\alpha)\varphi'(\alpha)^j : j = 0, 1, 2, \cdots\}$.*

Not surprisingly, this agrees with Theorem 1 of [16] which describes the spectra of compact weighted composition operators on $H^2(\beta_\kappa)$.

Proof. Since α is the fixed point of a non-automorphism of the unit disk, $|\varphi'(\alpha)| < 1$, and each eigenvector given by Theorem 9 corresponds to a different eigenvalue of this self-adjoint operator. Thus $\{g_j\}$ is an orthogonal sequence of eigenvectors. Because the functions in $H^2(\beta_\kappa)$ are analytic in a neighborhood of α, an easy power series argument shows the span of this sequence is dense in $H^2(\beta_\kappa)$. $\square$

We next look at the case $-1 + a_0^2 = a_1$, for which

$$\varphi(z) = a_0 + \frac{a_1 z}{1 - a_0 z} = a_0 + \frac{(-1 + a_0^2)z}{1 - a_0 z} = \frac{a_0 - z}{1 - a_0 z}$$

is an automorphism of the unit disk with $\varphi \circ \varphi(z) = z$. From this, we see that $W_{f,\varphi}^2 = W_{f\circ\varphi,\varphi\circ\varphi} = T_{f\circ\varphi}$, so it is worth looking at $W_{f,\varphi}^2$.

Lemma 11. *Suppose*

$$\varphi(z) = a_0 + \frac{a_1 z}{1 - a_0 z} \qquad \text{and} \qquad f(z) = c(1 - a_0 z)^{-\kappa} = cK_{\varphi(0)}(z)$$

If $0 < a_0 < 1$, $a_1 = -1 + a_0^2$, *and* $c = (1 - a_0^2)^{\kappa/2}$ *then* $W_{f,\varphi}$ *is an invertible isometry.*

Proof. Let g be in $H^2(\beta_\kappa)$. Then,

$$\begin{aligned}
(W_{f,\varphi}^2 g)(z) &= W_{f,\varphi}(W_{f,\varphi} g)(z) = W_{f,\varphi}(f \cdot g \circ \varphi)(z) \\
&= \left(f \cdot f \circ \varphi \cdot g(\varphi \circ \varphi)\right)(z) \\
&= f(z) \cdot f(\varphi(z)) \cdot g(\varphi(\varphi(z))) \\
&= \frac{c}{(1 - a_0 z)^\kappa} \frac{c}{(1 - a_0 \frac{a_0 - z}{1 - a_0 z})^\kappa} g(z) = \frac{c^2}{(1 - a_0^2)^\kappa} g(z) \\
&= g(z)
\end{aligned}$$

where the last equality holds by the choice of c in the hypothesis. The operator $W_{f,\varphi}$ is self-adjoint therefore $W_{f,\varphi}^2 = W_{f,\varphi}^* W_{f,\varphi} = W_{f,\varphi} W_{f,\varphi}^*$. Thus the above computation tells us that $W_{f,\varphi}^* W_{f,\varphi} = W_{f,\varphi} W_{f,\varphi}^*$ is the identity operator which means $W_{f,\varphi}$ is an invertible isometry. $\square$

In this case, the fixed points of φ are $(1 \pm \sqrt{1 - a_0^2})/a_0$ and it is clear that $\alpha = (1 - \sqrt{1 - a_0^2})/a_0$ is inside the open disk. Hence φ is an automorphism of the unit disk with a fixed point inside the open unit disk. In the previous section, note that the hypotheses of Lemma 8 and of Theorem 9 refer to φ having an interior fixed point and therefore apply to the case we are considering.

In complete analogy to Corollary 10, we characterize the Hermitian isometric weighted composition operators on $H^2(\beta_\kappa)$.

Corollary 12. *For* $\kappa \geq 1$, *let* $H^2(\beta_\kappa)$ *be the weighted Hardy space with kernel function* $K_w(z) = (1 - \overline{w}z)^{-\kappa}$. *Suppose* a_0, a_1, *and* c *are real numbers such that* $0 < a_0 < 1$, $a_1 = -1 + a_0^2$, *and* $c = (1 - a_0^2)^{\kappa/2}$. *If* φ, f, *and* g_j *are defined as in Theorem 9 and* α *is the fixed point of* φ *in the disk, then* $\{g_j\}_{j=0}^\infty$ *is an orthogonal basis for* $H^2(\beta_\kappa)$ *consisting of eigenvectors for* $W_{f,\varphi}$ *with* $W_{f,\varphi}(g_j) = (-1)^j g_j$. *In* *particular, $W_{f,\varphi}$ is Hermitian and unitary and the spectrum of $W_{f,\varphi}$ is the set $\{-1,1\}$.*

Proof. The map here is the map φ of Lemma 8 with $a_1 = -1+a_0^2$. From Lemma 8, we get $g(\varphi(z)) = \left(\dfrac{a_1 - a_0^2 + a_0\alpha}{1-a_0\alpha}\right)\dfrac{\alpha - z}{1-\alpha z}$. Because $a_1 - a_0^2 = -1$, it follows that $g(\varphi(z)) = (-1)\dfrac{\alpha - z}{1-\alpha z} = -g(z)$.

If j is a non-negative integer $g_j(z) = \dfrac{1}{(1-\alpha z)^\kappa}(g(z))^j$ where $g(z) = \dfrac{\alpha - z}{1-\alpha z}$. Then,

$$W_{f,\varphi}(g_j)(z) = \frac{(1-a_0^2)^{\kappa/2}}{(1-a_0 z)^\kappa}\frac{1}{(1-\alpha\varphi(z))^\kappa}(g(\varphi(z)))^j$$

using Lemma 8 we get,

$$W_{f,\varphi}(g_j)(z) = \frac{(1-a_0^2)^{\kappa/2}}{(1-a_0 z)^\kappa}\left(\frac{1-a_0 z}{(1-a_0\alpha)(1-\alpha z)}\right)^\kappa\left((-1)\frac{\alpha - z}{1-\alpha z}\right)^j$$

but $\alpha = \dfrac{1-\sqrt{1-a_0^2}}{a_0}$ hence $1 - a_0\alpha = \sqrt{1-a_0^2}$. Thus

$$W_{f,\varphi}(g_j)(z) = (-1)^j\frac{1}{(1-\alpha z)^\kappa}\left(\frac{\alpha - z}{1-\alpha z}\right)^j = (-1)^j g_j(z)$$

In Corollary 10, we showed that $\{g_j\}_{j=0}^\infty$ is an orthogonal basis for $H^2(\beta_\kappa)$, and for this reason, we know that the spectrum has no other elements besides ± 1. $\square$

3. Absolutely continuous Hermitian weighted composition operators

To study the remaining cases of Hermitian weighted composition operators, when $a_1 = (1-a_0)^2$ for $0 < a_0 < 1$, we first show that the operators $W_{f,\varphi}$ belong to a continuous semigroup of Hermitian operators. Recall that an indexed collection $\{A_t : t \geq 0\}$ of bounded operators is called a continuous semigroup of operators if $A_{s+t} = A_s A_t$ for all non-negative real numbers s and t, $A_0 = I$, and the map $t \mapsto A_t$ is strongly continuous. Similarly, for $0 < \theta \leq \frac{\pi}{2}$, an indexed collection $\{A_t : |\arg t| < \theta\}$ of bounded operators is called an analytic semigroup of operators if $A_{s+t} = A_s A_t$ for $|\arg s| < \theta$ and $|\arg t| < \theta$, the map $t \mapsto A_t$ is analytic in the angular domain $\{t : t \neq 0 \text{ and } |\arg t| < \theta\}$, and $\lim_{t\to 0, |\arg t|<\theta} A_t h = h$, in norm, for all h.

An easy calculation, presented in [11] and other sources, shows the collection of weighted composition operators $\{T_{f_t}C_{\varphi_t}\}$ satisfies the semigroup property if and only if

$$f_s \cdot (f_t \circ \varphi_s) = f_{s+t} \tag{9}$$

and

$$\varphi_t \circ \varphi_s = \varphi_{s+t} \tag{10}$$

In other words, the composition operator factors in a semigroup of weighted composition operators form a semigroup of composition operators and the Toeplitz operator factors form a 'cocycle' of Toeplitz operators. It is worthy of note that for each t with $\operatorname{Re} t \geq 0$, the linear fractional map φ_t maps the unit disk into itself, properly if $\operatorname{Re} t > 0$ and φ_t is an automorphism if t is on the imaginary axis and for each such t, $\varphi_t(1) = \varphi_t'(1) = 1$.

Let $\mathcal{P} = \{t : \operatorname{Re} t > 0\}$ denote the right half plane. For t in $\mathcal{P}$, let $A_t = T_{f_t} C_{\varphi_t}$ where

$$f_t(z) = \frac{1}{(1+t-tz)^\kappa} \tag{11}$$

and

$$\varphi_t(z) = \frac{t+(1-t)z}{1+t-tz} \tag{12}$$

Note that the relationship between a_0 and t can be expressed as $a_0 = \varphi_t(0) = t/(1+t)$ for $|a_0 - \frac{1}{2}| < \frac{1}{2}$. However, $a_1 = \varphi_t'(0) = (1+t)^{-2}$ is not real unless t is real, so A_t is not Hermitian on $H^2(\beta_\kappa)$ for t not real. It is somewhat tedious, but not difficult, to use the semigroup structure proved in Theorem 13 and the calculation $A_t^* = A_{\bar{t}}$ to show that the A_t are all normal operators, but we put this off to do it more easily later (Corollary 19).

Theorem 13. *The A_t, for t in the right half plane $\mathcal{P}$, form an analytic semigroup of weighted composition operators on $H^2(\beta_\kappa)$.*

Let Δ be the infinitesimal generator of the semigroup $\{A_t\}$. The domain of Δ is $\mathcal{D}_A = \{f \in H^2(\beta_\kappa) : (z-1)^2 f' \in H^2(\beta_\kappa)\}$. For such f,

$$\Delta(f)(z) = (z-1)^2 f'(z) + \kappa(z-1)f(z)$$

Proof. To show $A_t A_s = A_{t+s}$, it suffices to show that the cocycle relationship (Equation (9)) and the semigroup relationship (Equation (10)) hold. For the f_t and φ_t given above, the required equalities are

$$\begin{aligned} f_s(z) \cdot f_t(\varphi_s(z)) &= \frac{1}{(1+s-sz)^\kappa} \frac{1}{(1+t-t\frac{s+(1-s)z}{1+s-sz})^\kappa} \\ &= \frac{1}{(1+(s+t)-(s+t)z)^\kappa} = f_{s+t}(z) \end{aligned}$$

and

$$\varphi_s(\varphi_t(z)) = \frac{s+(1-s)\frac{t+(1-t)z}{1+t-tz}}{1+s-s\frac{t+(1-t)z}{1+t-tz}} = \frac{(s+t)+(1-(s+t))z}{1+(s+t)-(s+t)z} = \varphi_{s+t}(z)$$

Thus, the set $\{A_t : \operatorname{Re} t > 0\}$ is a semigroup of weighted composition operators.

Since operator valued functions are analytic in the norm topology if and only if they are analytic in the weak-operator topology (Theorem 3.10.1 of [18, p. 93]), it is sufficient to check that the map $t \mapsto \langle A_t h, K_z \rangle$ is analytic for each t in the right half plane. This is easy to see because h is analytic and

$$\langle A_t h, K_z \rangle = f_t(z) h(\varphi_t(z)) = \frac{1}{(1+t-tz)^\kappa} h\left(\frac{t+(1-t)z}{1+t-tz}\right)$$

which is clearly analytic in t for fixed z.

Finally, we must show the strong continuity at $t = 0$. First note that for $|t| \leq 1/3$, when $|z| < 1$,

$$|f_t(z)| = \frac{1}{|1 + t - tz|^\kappa} \leq \frac{1}{(1 - |t||1 - z|)^\kappa} \leq \frac{1}{(1 - 2/3)^\kappa} = 3^\kappa$$

so for $|t| \leq 1/3$, we have $\|f_t\|_\infty \leq 3^\kappa$. Similarly, the results of Exercises 2.1.5 and 3.1.3 of [9] imply that the norms $\|C_{\varphi_t}\|$ are uniformly bounded if $|t| \leq 1/3$ and t is in $\mathcal{P}$, so, we see that the norms $\|A_t\|$ are uniformly bounded on this set.

Observe that for α in the disk,

$$(A_t K_\alpha)(z) = f_t(z) K_\alpha(\varphi_t(z)) = \frac{1}{(1 + t - tz - \overline{\alpha}(t + (1 - t)z))^\kappa}$$

As t approaches 0, the functions $A_t K_\alpha$ converge uniformly, and therefore in $H^2(\beta_\kappa)$, to K_α. Since the kernel functions have dense span in $H^2(\beta_\kappa)$ and the norms $\|A_t\|$ are uniformly bounded for t in $\mathcal{P}$ with $|t| \leq 1/3$, it follows that for each f in $H^2(\beta_\kappa)$, we also have $\lim_{t \to 0, t \in \mathcal{P}} A_t f = f$. Thus, A_t is strongly continuous at $t = 0$ and the proof that $\{A_t\}$ is an analytic semigroup is complete.

To consider the infinitesimal generator, let f be a function in $H^2(\beta_\kappa)$ and suppose z is a point of the disk. Then we have

$$\begin{aligned}
(\Delta f)(z) &= \lim_{t \to 0^+} \frac{1}{t} \left((A_t - I) f \right)(z) = \lim_{t \to 0^+} \frac{f_t(z) f(\varphi_t(z)) - f(z)}{t} \\
&= \lim_{t \to 0^+} f_t(z) \frac{f(\varphi_t(z)) - f(z)}{\varphi_t(z) - z} \frac{\varphi_t(z) - z}{t} + \frac{f_t(z) - 1}{t} f(z) \\
&= 1 \cdot f'(z) \cdot (1 - z)^2 - \kappa(1 - z) f(z) = (z - 1)^2 f'(z) + \kappa(z - 1) f
\end{aligned}$$

This means that if f is in the domain $\mathcal{D}_A$ of Δ, then $(z - 1)^2 f'(z) + \kappa(z - 1) f(z)$ must be in $H^2(\beta_\kappa)$. Since the operator of multiplication by $z - 1$ is bounded on $H^2(\beta_\kappa)$, when f is in $H^2(\beta_\kappa)$, $(z - 1) f$ is also in $H^2(\beta_\kappa)$, so we conclude that if f is in the domain of Δ, then $(z - 1)^2 f' = \Delta(f) - (z - 1) f$ must be in $H^2(\beta_\kappa)$.

To complete the proof, we must show that if f is a function in $H^2(\beta_\kappa)$ such that $(z - 1)^2 f'$ is also in $H^2(\beta_\kappa)$, then f is in the domain of Δ. Clearly, every polynomial is in $\mathcal{D}_A$ because for a polynomial, the convergence of the limit in the calculation above is uniform on the closed disk, and therefore is a limit in the $H^2(\beta_\kappa)$ norm.

The remainder of the proof closely follows the proof of the determination of the domain of the infinitesimal generator in [11]. We very briefly outline and update that proof here.

Suppose f is a function in $H^2(\beta_\kappa)$ and suppose $(z - 1)^2 f'$ is also in $H^2(\beta_\kappa)$. Letting $f(z) = \sum_{k=0}^\infty a_k z^k$, a straightfoward calculation gives

$$(z - 1)^2 f'(z) = a_1 + (-2a_1 + 2a_2) z + \sum_{k=2}^\infty \left((k - 1) a_{k-1} - 2k a_k + (k + 1) a_{k+1} \right) z^k$$

Since $(z - 1)^2 f'$ is also in $H^2(\beta_\kappa)$, this series converges in $H^2(\beta_\kappa)$ norm to $(z - 1)^2 f'$.

For each positive integer n, let p_n be the polynomial

$$p_n(z) = \sum_{k=0}^{n} \frac{n-k}{n} a_k z^k$$

which is the polynomial approximation to f suggested by Cesáro summation. Then $\lim_{n\to\infty} p_n = f$ and the convergence is in the $H^2(\beta_\kappa)$ norm. In addition, we let r_n be the polynomial $r_n(z) = (z-1)^2 p_n'(z)$.

Now, let q_n be the polynomial

$$q_n(z) = a_1 + \frac{n-1}{n}(-2a_1 + 2a_2)z + \sum_{k=2}^{n} \frac{n-k}{n} \left((k-1)a_{k-1} - 2ka_k + (k+1)a_{k+1}\right) z^k$$

which is the polynomial approximation to $(z-1)^2 f$ suggested by Cesáro summation. Then $\lim_{n\to\infty} q_n = (z-1)^2 f'$ and the convergence is in the $H^2(\beta_\kappa)$ norm.

In [11], it was proved that $\lim_{n\to\infty} \|q_n - r_n\|_{H^2} = 0$. The norm of any polynomial p in $H^2(\beta_\kappa)$ satisfies $\|p\|_{H^2(\beta_\kappa)} \le \|p\|_{H^2}$, so we have $\lim_{n\to\infty} \|q_n - r_n\|_{H^2(\beta_\kappa)} = 0$. Since $\lim_{n\to\infty} q_n = (z-1)^2 f'$, this means that $\lim_{n\to\infty} r_n = (z-1)^2 f'$ also.

Now, $\lim_{n\to\infty} p_n = f$ which implies, because multiplication by $z-1$ is bounded, that $\lim_{n\to\infty}(z-1)p_n = (z-1)f$. We note that $\Delta(p_n) = (z-1)^2 p_n' + \kappa(z-1)p_n = r_n + \kappa(z-1)p_n$ which means that $\lim_{n\to\infty} \Delta(p_n) = \lim_{n\to\infty} r_n + \lim_{n\to\infty} \kappa(z-1)p_n = (z-1)^2 f' + \kappa(z-1)f$.

Thus, we have $\lim_{n\to\infty} p_n = f$ and $\lim_{n\to\infty} \Delta(p_n) = (z-1)^2 f' + \kappa(z-1)f$. Since Δ is a closed operator, this means that f is in $\mathcal{D}_A$, the domain of Δ, and that $\Delta(f) = (z-1)^2 f' + \kappa(z-1)f$. □

We want to find the potential eigenvectors and eigenvalues of the infinitesimal generator and consider their contribution to the spectra of Δ and the A_t.

Lemma 14. *If f is analytic in the disk and $(z-1)^2 f' + \kappa(z-1)f = \lambda f(z)$, then*

$$f(z) = f_\lambda(z) = \frac{C}{(1-z)^\kappa} e^{\frac{\lambda}{1-z}}$$

for some constant C.

Proof. If $(z-1)^2 f' + \kappa(z-1)f = \lambda f(z)$, then

$$(z-1)^2 f'(z) = \left(-\kappa(z-1) + \lambda\right) f(z)$$

Thus

$$\frac{f'(z)}{f(z)} = -\frac{\kappa}{z-1} + \frac{\lambda}{(z-1)^2}$$

Integrating both sides, we get

$$\ln f(z) = -\kappa \ln(z-1) - \lambda \frac{1}{z-1} + c$$

for some constant c. This implies

$$f(z) = \frac{C}{(1-z)^\kappa} e^{\frac{\lambda}{1-z}}$$

for some constant C. □

Corollary 15. *For $\kappa \geq 1$, the functions f_λ of Lemma 14 do not belong to $H^2(\beta_\kappa)$, and for $t > 0$, the operator A_t has no eigenvalues.*

Proof. Because $\{A_t\}_{t\geq 0}$ is a strongly continuous semigroup of bounded operators,

$$\sigma_p(A_t) \subset e^{t\sigma_p(\Delta)} \cup \{0\}$$

and specifically, if λ is in $\sigma_p(\Delta)$, then $e^{\lambda t}$ is in $\sigma_p(A_t)$. (See [22, p. 46], for example.) Because $A_t = T_{f_t} C_{\varphi_t}$ and both the Toeplitz operator and the composition operator are one-to-one, 0 is not in the point spectrum of A_t. From Lemma 14, we know that λ is in $\sigma_p(\Delta)$ if and only if

$$f_\lambda(z) = \frac{1}{(1-z)^\kappa} e^{\frac{\lambda}{1-z}} \in H^2(\beta_\kappa)$$

and if $f_\lambda(z)$ is in $H^2(\beta_\kappa)$ then the above containment means that $e^{\lambda t}$ is an eigenvalue of A_t. Now for $t > 0$, the operators A_t are Hermitian and this means any eigenvalues must be real and $e^{\lambda t}$ real for all $t > 0$ implies λ is real. In this case, if $\nu < \lambda$, then $|f_\nu(z)| \leq |f_\lambda(z)|$ for all z in $\mathbb{D}$, so f_ν is also in $H^2(\beta_\kappa)$ and $e^{\nu t}$ is also an eigenvalue of A_t. For self-adjoint operators on a Hilbert space, the eigenvectors corresponding to distinct eigenvalues are orthogonal. Since the number of real numbers less than λ is uncountable and $H^2(\beta_\kappa)$ is a separable Hilbert space, it is not possible for $e^{\nu t}$ to be an eigenvalue for every $\nu < \lambda$, so it is not possible for $e^{\lambda t}$ to be an eigenvalue for any real number λ. It follows that no f_λ is in $H^2(\beta_\kappa)$, and that the operators A_t have no eigenvalues. □

Besides being part of an analytic semigroup, these Hermitian weighted composition operators have another special property: each of these operators is cyclic on $H^2(\beta_\kappa)$.

Theorem 16. *For $\kappa \geq 1$, let $H^2(\beta_\kappa)$ be the weighted Hardy space with kernel functions $K_w(z) = (1 - \overline{w}z)^{-\kappa}$. If $W_{f,\varphi}$ is the Hermitian weighted composition operator on $H^2(\beta_\kappa)$ with*

$$f(z) = (1 - a_0 z)^{-\kappa} \qquad \text{and} \qquad \varphi(z) = a_0 + \frac{a_1 z}{1 - a_0 z}$$

where $0 < a_0 < 1$ and $a_1 = (1 - a_0)^2$, then $W_{f,\varphi}$ is a (star) cyclic Hermitian operator, indeed, the vector 1 in $H^2(\beta_\kappa)$ is a (star) cyclic vector for $W_{f,\varphi}$.

Proof. We note that because $W_{f,\varphi}$ is Hermitian, a vector is a star-cyclic vector exactly when it is a cyclic vector.

Since $0 < a_0 < 1$ and $a_1 = (1 - a_0)^2$, for $t = a_0/(1 - a_0)$ we have $W_{f,\varphi} = A_t$ in the notation above. For w in the unit disk,

$$A_t K_w = C_{\varphi_t}^* T_{f_t}^* K_w = \overline{f_t(w)} K_{\varphi_t(w)} = f_t(\overline{w}) K_{\varphi_t(w)}$$

Since the vector 1 in $H^2(\beta_\kappa)$ is K_0, we have $A_t(1) = f_t(0) K_{\varphi_t(0)}$. Now,

$$A_t\left(f_t(0) K_{\varphi_t(0)}\right) = A_t(A_t(1)) = A_{2t}(1) = f_{2t}(0) K_{\varphi_{2t}(0)}$$

and in general, clearly, $A_t^n(1) = f_{nt}(0) K_{\varphi_{nt}(0)}$.

To check cyclicity, we need to investigate the span of these vectors. Since the factor $f_{nt}(0)$ is just a non-zero number, 1 is a cyclic vector for A_t if and only if span$\{K_{\varphi_{nt}(0)}\}$ is dense in $H^2(\beta_\kappa)$. This span is dense if and only if 0 is the only vector orthogonal to all the vectors $K_{\varphi_{nt}(0)}$. Since $\langle h, K_{\varphi_{nt}(0)}\rangle = h(\varphi_{nt}(0))$, this means that the span is dense if and only if the only function h in $H^2(\beta_\kappa)$ such that $h(\varphi_{nt}(0)) = 0$ for $n = 1, 2, 3, \cdots$ is the zero function. Note that since t is real and positive, each of the points $\varphi_{nt}(0) = nt/(1+nt)$ is in the interval $[0, 1)$ on the real axis.

Because $K_w(z) = (1-\overline{w}z)^{-\kappa}$, we see $\|K_w\| = (1-|w|^2)^{-\kappa/2}$ and by the Cauchy-Schwartz inequality, for each h in $H^2(\beta_\kappa)$, there is a constant C so that for all w in the unit disk,

$$|h(w)| = |\langle h, K_w\rangle| \leq \|h\|\|K_w\| = \frac{\|h\|}{(1-|w|^2)^{\kappa/2}} \leq C\exp\{(1-|w|)^{-.2}\}$$

This means that every function in $H^2(\beta_\kappa)$ satisfies the growth condition in the hypothesis of a theorem of Shapiro and Shields [25] (or see [12, p. 116]) on zero sets of analytic functions on the disk. It follows from this theorem that the set of points $\{\varphi_{nt}(0)\}$ is the zero set of a non-zero function in $H^2(\beta_\kappa)$ if and only if the sequence is a Blaschke sequence. To check this, consider the sum

$$\sum_{n=1}^{\infty}(1-|\varphi_{nt}(0)|) = \sum_{n=1}^{\infty}\left(1-\frac{nt}{1+nt}\right) = \sum_{n=1}^{\infty}\frac{1}{1+nt} = \infty$$

Since this sum is infinite, the sequence is *not* a Blaschke sequence and therefore the vectors $\{K_{\varphi_{nt}(0)}\}$ have dense span in $H^2(\beta_\kappa)$. This means 1 is a cyclic vector for $A_t = W_{f,\varphi}$. □

We will see that the estimate used in the proof of Theorem 13 above for $\|A_t\|$ can be improved: in fact, it follows from Theorem 17 and Corollary 19 that $\|A_t\| = 1$ for all t in $\mathcal{P}$.

Theorem 16 says that, for $t > 0$ (i.e. $0 < a_0 \leq 1$), each of the operators A_t is cyclic, which means that each is unitarily equivalent to an ordinary multiplication operator (see, for example, [2, p. 269]). The following result gives this explicitly. The results here are similar to those in the Hardy space case as presented in [11], but the strategy here will be to use the weighted composition operators to discover facts about the invariant subspaces in the weighted Bergman spaces, rather than using facts about invariant subspaces of the Hardy space to get properties of the operators A_t.

The relevant L^2 space is $L^2([0,1],\mu)$ where μ is the absolutely continuous probability measure [14, p. 581] given by

$$d\mu = \frac{(\ln(1/x))^{\kappa-1}}{\Gamma(\kappa)}\,dx$$

In the theorem below, the multiplication operator M_h is defined as usual: $(M_h g)(x) = h(x)g(x)$ for g in $L^2([0,1],\mu)$.

Theorem 17. *For each t in $\mathcal{P}$, the multiplication operator M_{x^t} on $L^2([0,1],\mu)$ and the weighted composition operator A_t on $H^2(\beta_\kappa)$ are unitarily equivalent. In fact, the operator $U : H^2(\beta_\kappa) \to L^2$ given by, for s in $\mathcal{P}$,*

$$U(f_s) = x^s$$

is unitary and satisfies $UA_t = M_{x^t}U$.

Proof. First note that functions in $L^\infty([0,1])$ are in $L^2([0,1],\mu)$ so the Lebesgue Dominated Convergence Theorem shows that uniform convergence on $[0,1]$ implies norm convergence in $L^2([0,1],\mu)$. It also follows that if h is in $L^\infty([0,1])$ then M_h is a bounded operator on this Hilbert space with norm $\|M_h\| = \|h\|_\infty$.

The Stone-Weierstrass Approximation Theorem shows that $\{x^n\}_{n=0,1,2,\cdots}$, has dense span in $L^2([0,1],\mu)$. In particular, this means the span of $\{x^s : s \in \mathcal{P}\}$ is dense in $L^2([0,1],\mu)$.

In the proof of Theorem 16, we saw that

$$f_n(z) = f_n(0)K_{\varphi_n(0)}(z)$$

for $n = 0,1,2,\cdots$, has dense span in $H^2(\beta_\kappa)$, so the span of $\{f_s : s \in \mathcal{P}\}$ is dense in $H^2(\beta_\kappa)$.

Thus, if U satisfies $U(f_s) = x^s$, we see that for each s in $\mathcal{P}$, using Equation (9), we have

$$UA_t(f_s) = Uf_{s+t} = x^{s+t} = M_{x^t}x^s = M_{x^t}U(f_s)$$

Since the span of $\{f_s : s \in \mathcal{P}\}$ is dense in $H^2(\beta_\kappa)$, we conclude $UA_t = M_{x^t}U$.

We will show that U is isometric on the span of $\{f_s\}$ for s in $\mathcal{P}$. For r and s in $\mathcal{P}$, we have

$$\begin{aligned}
\langle f_r, f_s\rangle &= \langle f_r(z), \left(\frac{1}{1+s-sz}\right)^\kappa\rangle = \langle f_r(z), \left(\frac{1}{1+s}\right)^\kappa \left(\frac{1}{1-\frac{s}{1+s}z}\right)^\kappa\rangle \\
&= \overline{\left(\frac{1}{1+s}\right)^\kappa}\langle f_r(z), K_{\frac{s}{1+s}}(z)\rangle \\
&= \overline{\left(\frac{1}{1+s}\right)^\kappa}\left(\frac{1}{1+r-r\overline{\frac{s}{(1+s)}}}\right)^\kappa \\
&= \frac{1}{((1+r)(1+\bar{s})-r\bar{s})^\kappa} \\
&= \frac{1}{(1+\bar{s}+r)^\kappa}.
\end{aligned}$$

Similarly, for r and s in $\mathcal{P}$, in $L^2([0,1],\mu)$, we have

$$\begin{aligned}
\langle x^r, x^s\rangle_{L^2} &= \int_0^1 x^r\overline{x^s}\frac{(-\ln x)^{\kappa-1}}{\Gamma(\kappa)}\,dx = \frac{1}{\Gamma(\kappa)}\int_0^1 x^{r+\bar{s}}(-\ln x)^{\kappa-1}\,dx \\
&= \frac{1}{\Gamma(\kappa)}\frac{\Gamma(\kappa)}{(1+\bar{s}+r)^\kappa} \\
&= \frac{1}{(1+\bar{s}+r)^\kappa}.
\end{aligned}$$

(When κ is a positive integer the integral above can be evaluated by applying integration by parts repeatedly but the general integral can be found in integral tables, for example, see [14, p. 581].)

Now, the vectors $\{f_s\}$ have dense span in $H^2(\beta_\kappa)$ and the vectors $\{x^s\}$ have dense span in $L^2([0,1],\mu)$. The inner product calculations above show that U is isometric on these spans and therefore has a unique extension to an isometric operator of $H^2(\beta_\kappa)$ onto $L^2([0,1],\mu)$, that is, it is a unitary operator between these spaces. $\square$

Corollary 18. *For each t with $t > 0$, the Hermitian weighted composition operator A_t is unitarily equivalent to M_{x^t}. In particular, for each $t > 0$, these operators satisfy $\|A_t\| = 1$ and have spectrum $\sigma(A_t) = [0,1]$.*

Proof. For each t with $0 < t < \infty$, Theorem 17 implies $A_t = U^* M_{x^t} U$ so

$$\|A_t\| = \|M_{x^t}\| = \sup\{|x^t| : 0 \le x \le 1\} = 1$$

and

$$\sigma(A_t) = \sigma(M_{x^t}) = \{x^t : 0 \le x \le 1\} = [0,1]$$

$\square$

Corollary 19. *For each t in the right half plane $\mathcal{P}$, the weighted composition operator A_t is unitarily equivalent to M_{x^t}. In particular, for each t in the right half plane $\mathcal{P}$, the weighted composition operator A_t is normal and satisfies $A_t^* = A_{\bar{t}}$. Moreover, for $t = a + ib$ with $a > 0$, these operators satisfy $\|A_t\| = 1$ and have spectrum that is a spiral, $\sigma(A_t) = \{0\} \cup \{e^{ay}e^{iby} : y \le 0\}$.*

Proof. For each t in the right half plane $\mathcal{P}$, Theorem 17 implies $A_t = U^* M_{x^t} U$. Multiplication operators M_h are normal for every h in $L^\infty([0,1])$ and satisfy $M_h^* = M_{\overline{h}}$. Since unitary equivalence preserves normality and adjoints, we have

$$A_t^* = (U^* M_{x^t} U)^* = U^* M_{x^t}^* U = U^* M_{x^{\bar{t}}} U = A_{\bar{t}}$$

and

$$\|A_t\| = \|M_{x^t}\| = \sup\{|x^t| : 0 \le x \le 1\} = \sup\{x^a : 0 \le x \le 1\} = 1$$

and, letting $x = e^y$ for $0 < x \le 1$

$$\sigma(A_t) = \sigma(M_{x^t}) = \{x^t : 0 \le x \le 1\} = \{0\} \cup \{e^{yt} : y \le 0\} = \{0\} \cup \{e^{ay}e^{iby} : y \le 0\}$$

$\square$

As we would expect, from the Hermitian and the non-Hermitian, normal cases, t imaginary corresponds to A_t being unitary. Such weighted composition operators have been considered by Forelli [13] who showed that they are isometries of H^2 and Bourdon and Narayan [1] who showed that they are unitary on H^2.

Corollary 20. *For each t imaginary, the weighted composition operator A_t is unitarily equivalent to M_{x^t}, which is a unitary operator. In particular, for each t on the imaginary axis, the weighted composition operator A_t is unitary and satisfies $A_t^* = A_{\bar{t}} = A_{-t}$. Moreover, for such t, these operators have spectrum the unit circle, $\sigma(A_t) = \{e^{iy} : y \le 0\}$.*

Proof. The calculations in the proof of Theorem 13 that show $A_s A_t = A_{s+t}$ do not depend on $\mathrm{Re}\, t > 0$, so the equation $A_t^* = A_{\bar{t}} = A_{-t}$ shows that $A_t^* A_t = A_t A_t^* = I$ and A_t is unitary for t on the imaginary axis.

For each t in the right half plane $\mathcal{P}$, Theorem 17 says $UA_t = M_{x^t}U$. Writing $t = a + ib$ and taking weak operator limits we see that

$$UA_{ib} = \operatorname*{w\text{-}lim}_{a\to 0^+} UA_{a+ib} = \operatorname*{w\text{-}lim}_{a\to 0^+} M_{x^{a+ib}}U = M_{x^{ib}}U$$

so A_t and M_{x^t} are unitarily equivalent also for t on the imaginary axis. The spectral result is correct because $\sigma(M_{x^t}) = \{x^t : 0 < x \le 1\}$. □

In the case of the usual Hardy space, H^2, the spectral subspaces for the weighted composition operators were [11] the subspaces χH^2 where χ is an atomic singular inner function with atom at $\{1\}$. Our goal will be to prove the analogue of this result for the weighted Bergman spaces $H^2(\beta_\kappa) = A^2_{\kappa-2}$.

We define subspaces H_c of $H^2(\beta_\kappa) = A^2_{\kappa-2}$ as follows: Let $H_0 = H^2(\beta_\kappa)$. For c a negative real number, define the subspace H_c by

$$H_c = \text{ closure } \{e^{c\frac{1+z}{1-z}} f : f \in H^2(\beta_\kappa)\} \tag{13}$$

These subspaces are well known and have been studied because they are relatively simple invariant subspaces for multiplication by z on $H^2(\beta_\kappa)$ that are not associated with a zero set. The properties of these subspaces that we need are summarized in Lemma 21 below, but we will not prove it. Good sources for this kind of result are the books of Hedenmalm, Korenblum, and Zhu [17], especially Section 3.2, pp. 55-59, or Duren and Schuster [12], Section 8.2, where proofs are given for the usual (unweighted) Bergman space. Other sources are Theorem 2 of Roberts [23] or Korenblum [19].

Lemma 21. *For each negative number c, the subspace H_c is a proper subspace of $H^2(\beta_\kappa)$ and if c_1 and c_2 are negative real numbers with $c_2 < c_1$, then H_{c_2} is a proper subspace of H_{c_1}. Finally, $\cap_{c\le 0} H_c = (0)$.*

The idea underlying the focus on these subspaces is that the eigenvectors of our operators, if they had any eigenvectors, should be

$$\frac{1}{(1-z)^\kappa} e^{\frac{\lambda}{1-z}} = \frac{e^{\frac{\lambda}{2}}}{(1-z)^\kappa} e^{\frac{\lambda}{2}\frac{1+z}{1-z}}$$

If these were eigenvectors for A_1, say, then the spectral subspace associated with $[0, r]$ for $0 \le r \le 1$ would be spanned by the eigenvectors whose eigenvalues are in $[0, r]$. This will be the case for A_1 if and only if λ is a number so that $0 < e^\lambda \le r$. So suppose r is given and $\lambda_0 = \ln r$ so that $e^{\lambda_0} = r$. Looking at the "eigenvectors", it looks like the subspace containing the eigenvectors for the eigenvalues with $0 < e^\lambda < r$ ought, if they were actually in $H^2(\beta_\kappa)$, to span the subspace $e^{\frac{\lambda_0}{2}\frac{1+z}{1-z}} H^2(\beta_\kappa) = H_{(\ln r)/2}$.

Theorem 22 below shows directly that these are invariant subspaces for the A_t. We want to prove that these are, indeed, the spectral subspaces we are looking for. In [11], this was accomplished for the Hardy space by using the special properties of these well-studied subspaces to identify the projections and construct the spectral integrals. However, in the context of $H^2(\beta_\kappa) = A^2_{\kappa-2}$, these subspaces are not as well understood as in the Hardy space. Instead, we will reverse the strategy: we will use the unitary U to carry information from the multiplication operators on $L^2([0, 1], \mu)$, which are well understood, back to the weighted composition operators on $H^2(\beta_\kappa)$, and use this information to better understand these invariant subspaces of the Bergman spaces.

Theorem 22. *For $0 \le t$ and $c \le 0$, the subspace H_c is invariant for A_t.*

Proof. Suppose f is in $H^2(\beta_\kappa)$ so that $(\exp(c(1+z)/(1-z)))\,f$ is in H_c. Then

$$A_t\left(e^{c\frac{1+z}{1-z}}f\right) = \frac{1}{(1+t-tz)^\kappa}\left(e^{c\frac{1+\frac{t+(1-t)z}{1+t-tz}}{1-\frac{t+(1-t)z}{1+t-tz}}}\right) f\circ\varphi_t$$

$$= \frac{1}{(1+t-tz)^\kappa}\left(e^{c\left(2t+\frac{1+z}{1-z}\right)}\right) f\circ\varphi_t = e^{c\frac{1+z}{1-z}}\frac{e^{2ct}}{(1+t-tz)^\kappa} f\circ\varphi_t$$

$$= e^{c\frac{1+z}{1-z}}A_t(e^{2ct}f)$$

and we see this latter vector is also in H_c. Since

$$\{(\exp(c(1+z)/(1-z)))\,f : f \in H^2(\beta_\kappa)\}$$

is dense in H_c and H_c is closed, this shows that H_c is invariant for A_t. □

For $t > 0$, the functions x^t are increasing for $0 \le x \le 1$. This means the spectral subspaces for the (Hermitian) multiplication operators M_{x^t} are just the subspaces $L_\delta = \{f \in L^2([0,1],\mu) : f(x) = 0 \text{ for } \delta < x \le 1\}$, for $0 \le \delta \le 1$. Obviously, $L_0 = (0)$, $L_1 = L^2$, for $\delta_1 < \delta_2$, the subspaces satisfy $L_{\delta_1} \subset L_{\delta_2}$, and the orthogonal projection of L^2 onto L_δ is the multiplication operator $M_{\chi_{[0,\delta]}}$. We want to identify U^*L_δ in $H^2(\beta_\kappa)$.

Theorem 23. *For $c < 0$ and $0 < \delta < 1$, let H_c and L_δ be as above. If U is the unitary operator of $H^2(\beta_\kappa)$ onto $L^2([0,1],\mu)$ as in Theorem 17, then*

$$U^*L_\delta = H_{(\ln\delta)/2} \quad \textit{or equivalently} \quad UH_c = L_{e^{2c}}$$

Proof. Clearly the two versions of the conclusion are equivalent; we will prove the first. Since this result follows directly from the work of [11] for H^2 when $\kappa = 1$, we may assume that $\kappa > 1$. For $0 < a < 1$, let f_a be the function defined by

$$f_a(x) = \begin{cases} (\ln(1/x))^{-\kappa+1} & 0 < x \le a \\ 0 & a < x \le 1 \end{cases}$$

It is not difficult to see that $\lim_{x\to 0^+} f_a(x) = 0$, that f_a is continuous and increasing on $(0,a]$ and that therefore f_a is in $L^\infty([0,1])$ and also in $L^2([0,1],\mu)$.

For $t > 0$, computing in L^2, we have

$$\langle f_a, x^t\rangle = \frac{1}{\Gamma(\kappa)}\int_0^a \frac{1}{(\ln(1/x))^{\kappa-1}}x^t d\mu(x) = \frac{1}{\Gamma(\kappa)}\int_0^a x^t dx = \frac{1}{\Gamma(\kappa)}\frac{a^{t+1}}{t+1}$$

Let $F_a = U^*f_a$, and computing in $H^2(\beta_\kappa)$, we have

$$\begin{aligned}\frac{1}{\Gamma(\kappa)}\frac{a^{t+1}}{1+t} &= \langle f_a, x^t\rangle = \langle U^*f_a, U^*x^t\rangle = \langle F_a, \frac{1}{(1+t-tz)^\kappa}\rangle \\ &= \frac{1}{(1+t)^\kappa}\langle F_a, \frac{1}{(1-\frac{t}{1+t}z)^\kappa}\rangle = \frac{1}{(1+t)^\kappa}F_a(\frac{t}{1+t})\end{aligned}$$

where the last equality holds because t is positive exactly when $t/(1+t)$ is a point of $(0,1)$ in the unit disk and the inner product is the point evaluation of F_a.

Rewriting the equality, we have, for every $t > 0$,

$$F_a\left(\frac{t}{1+t}\right) = \frac{1}{\Gamma(\kappa)}(1+t)^{\kappa-1}a^{t+1}$$

Since F_a is analytic in the disk, the equality for all $t > 0$ gives equality whenever $z = \frac{t}{1+t}$ is in the disk. In particular, we find

$$F_a(z) = \frac{1}{\Gamma(\kappa)}\frac{1}{(1-z)^{\kappa-1}}e^{\frac{\ln a}{1-z}} = \frac{e^{(\ln a)/2}}{\Gamma(\kappa)}\frac{1}{(1-z)^{\kappa-1}}e^{((\ln a)/2)(\frac{1+z}{1-z})}$$

If $s(x)$ is a simple function defined on $[0, 1]$ such that $s^{-1}(\{y\})$ is an open interval for each y in the range of s, we can easily approximate s in L^∞ with linear combinations of functions of the form $f_a - f_b$ for $0 \le b < a \le 1$. It follows that the closure in $L^2([0, 1], \mu)$ of the span of $\{f_a : 0 < a \le \delta\}$ is L_δ.

Thus, we have

$$\begin{aligned} U^*(L_\delta) &= \overline{\text{span}}\{U^* f_a : 0 < a \le \delta\} \\ &= \overline{\text{span}}\left\{\frac{e^{(\ln a)/2}}{\Gamma(\kappa)}\frac{1}{(1-z)^{\kappa-1}}e^{((\ln a)/2)(\frac{1+z}{1-z})} : 0 < a \le \delta\right\} \\ &= \overline{\text{span}}\left\{\frac{1}{(1-z)^{\kappa-1}}e^{((\ln a)/2)(\frac{1+z}{1-z})} : 0 < a \le \delta\right\} \\ &= H_{(\ln \delta)/2} \end{aligned}$$

□

For $0 < r \le 1$, let P_r be the orthogonal projection of $H^2(\beta_\kappa)$ onto the subspace $H_{(\ln r)/2}$ and let $P_0 = 0$. The set of projections $\{P_r\}_{0 \le r \le 1}$ form a resolution of the identity and each commutes with each A_t because each A_t is Hermitian and each $H_{(\ln r)/2}$ is invariant for A_t (Theorem 22).

Corollary 24. *Let t be a positive real number. Letting $P_0 = 0$, the projections $\{P_r\}_{0 \le r \le 1}$ form a resolution of the identity for the operator A_t. Related to A_t, the projection P_r corresponds to the interval $[0, r^t]$ as a subset of the spectrum of A_t. This means we have*

$$A_t = \int_0^1 r^t \, dP_r$$

Proof. For each t with $t > 0$, the subspace $L_\delta = M_{\chi_{[0,\delta]}} L^2$ is associated with $\{x : 0 \le x^t \le \delta\}$, so $M_{x^t} = \int_0^1 \delta^t \, dM_{\chi_{[0,\delta]}}$. The unitary U implements an equivalence between A_t and M_{x^t}, so L_δ is associated with $H_{(\ln r)/2}$ and $A_t = \int_0^1 r^t \, dP_r$. □

4. Extremal functions for invariant subspaces determined by atomic singular inner functions

Suppose N is a subspace of the Bergman space $A^2_{\kappa-2} = H^2(\beta_\kappa)$ that is invariant for the operator of multiplication by z. If there are functions f in N so that $f(0) \ne 0$ and G is a function of N so that

$$\|G\| = 1 \quad \text{and} \quad G(0) = \sup\{\text{Re}\, f(0) : f \in N \text{ and } \|f\| = 1\} \tag{14}$$

then we say G *solves the extremal problem for the invariant subspace* N and we say G *is an* $A^2_{\kappa-2}$ *inner function.* (For more information about such extremal problems, see [17, p. 56] or [12, p. 120 or pp. 146-152], for example.)

The subspaces H_c defined by Equation (13) for $c < 0$ are of interest to us because they are the spectral subspaces for the operators A_t, but they are of wider interest because they are invariant for multiplication by z and they are associated with the atomic singular inner functions $e^{c(1+z)/(1-z)}$ for $c < 0$. In the study of invariant subspaces for the Bergman shifts, it is important to solve the extremal problem of Equation (14), but it is not always straightforward to do so. We show that the unitary equivalence of Theorem 17 can be used to solve these problems for the subspaces H_c. Note first that $f(z) = e^{c(1+z)/(1-z)}$ is in H_c and $f(0) = e^c \neq 0$, so the formulation of the extremal problem given above is applicable.

Our computation of the extremal functions requires the use of the *incomplete Gamma function* [14, p. 949] usually defined as

$$\Gamma(a,w) = \int_w^\infty t^{a-1} e^{-t}\, dt \tag{15}$$

where a is a complex parameter and w is a real parameter. An alternate definition [14, p. 950] in which both a and w are complex parameters is

$$\Gamma(a,w) = e^{-w} w^a \int_0^\infty e^{-wu} (1+u)^{a-1}\, du \tag{16}$$

In our situation, we have $a \geq 0$ and for this case, we see that $\Gamma(a,w)$ is analytic for $\operatorname{Re} w > 0$.

Theorem 25. *For $c < 0$, if H_c is the invariant subspace of the weighted Bergman space $A^2_{\kappa-2} = H^2(\beta_\kappa)$ defined by*

$$H_c = \text{closure}\{e^{c\frac{1+z}{1-z}} f : f \in A^2_{\kappa-2} = H^2(\beta_\kappa)\}$$

then the extremal function for H_c is

$$G_c(z) = \frac{\Gamma(\kappa, -2c/(1-z))}{\sqrt{\Gamma(\kappa)}\sqrt{\Gamma(\kappa,-2c)}} \tag{17}$$

Proof. The Cauchy-Schwarz inequality implies that to maximize $\operatorname{Re}\langle g, f\rangle$ where f is given and g is a unit vector in the subspace N, then $g = \zeta Pf/\|Pf\|$ where P is the orthogonal projection onto N and ζ is chosen with $|\zeta| = 1$ so that $\langle g, f\rangle > 0$.

Suppose g is in H_c. Then $\operatorname{Re} g(0) = \operatorname{Re}\langle g, K_0\rangle = \operatorname{Re}\langle g, f_0\rangle$, where $f_0 = f_s$ for $s = 0$ as in Equation (11). Let G_c be the function in H_c that solves this extremal problem, that is, $G_c = \zeta P_{e^{2c}} f_0/\|P_{e^{2c}} f_0\|$ for ζ chosen with $|\zeta| = 1$ so that $G_c(0) > 0$. We want to find an explicit description of G_c. We will ignore ζ for the moment, and at the end of the proof, we will find an appropriate choice for ζ.

We want to move the problem to L^2; we have $\langle g, f_0\rangle = \langle Ug, Uf_0\rangle = \langle Ug, x^0\rangle = \langle Ug, 1\rangle$ and g is in H_c if and only if Ug is in $L_{e^{2c}}$. Writing δ for e^{2c}, we see that $UG_c = Q_\delta(1)/\|Q_\delta(1)\|$ where Q_δ is the projection of $H^2(\beta_\kappa)$ onto L_δ, that is, $Q_\delta = M_{\chi_{[0,\delta]}}$. It follows that

$$UG_c = \frac{1}{\|\chi_{[0,\delta]}\|}\chi_{[0,\delta]}$$

Now, to find the necessary norm,

$$\begin{aligned} \|P_{e^{2c}}f_0\|^2 &= \|\chi_{[0,\delta]}\|^2 = \int_0^1 |\chi_{[0,\delta]}|^2\,d\mu = \int_0^\delta d\mu \\ &= \int_0^\delta \frac{(\ln(1/x))^{\kappa-1}}{\Gamma(\kappa)}\,dx = \frac{1}{\Gamma(\kappa)}\int_{-2c}^\infty y^{(\kappa-1)}e^{-y}\,dy \\ &= \frac{\Gamma(\kappa,-2c)}{\Gamma(\kappa)} \end{aligned} \tag{18}$$

where the penultimate equality comes from the change of variables $x = e^{-y}$ and the last equality is the definition of the incomplete Gamma function [14, p. 949].

Similarly, to find the function itself, for $s > 0$, we have

$$\begin{aligned} \|P_{e^{2c}}f_0\|\langle G_c, f_s\rangle &= \|P_{e^{2c}}f_0\|\langle UG_c, Uf_s\rangle = \langle \chi_{[0,\delta]}, x^s\rangle = \int_0^\delta x^s\,d\mu \\ &= \int_0^\delta x^s\frac{(\ln(1/x))^{\kappa-1}}{\Gamma(\kappa)}\,dx = \frac{1}{\Gamma(\kappa)}\int_{-2c}^\infty y^{(\kappa-1)}e^{-(s+1)y}\,dy \\ &= \frac{1}{(s+1)^\kappa\Gamma(\kappa)}\int_{-2c(s+1)}^\infty u^{(\kappa-1)}e^{-u}\,du = \frac{\Gamma(\kappa,-2c(s+1))}{(s+1)^\kappa\Gamma(\kappa)} \end{aligned} \tag{19}$$

On the other hand, we can find the value of $\langle G_c, f_s\rangle$ another way because

$$f_s(z) = \frac{1}{(1+s-sz)^\kappa} = \frac{1}{(1+s)^\kappa}\frac{1}{(1-s/(s+1)z)^\kappa} = \frac{1}{(1+s)^\kappa}K_{s/(s+1)}$$

where K_α is the kernel for evaluation at α. Using this, we see

$$\langle G_c, f_s\rangle = \langle G_c, \frac{1}{(1+s)^\kappa}K_{s/(s+1)}\rangle = \frac{1}{(1+s)^\kappa}\langle G_c, K_{s/(s+1)}\rangle = \frac{G_c(s/(s+1))}{(1+s)^\kappa} \tag{20}$$

Putting the results of Equations (18), (19), and (20) together, we get

$$\|P_{e^{2c}}f_0\|\frac{G_c(s/(s+1))}{(1+s)^\kappa} = \|P_{e^{2c}}f_0\|\langle G_c, f_s\rangle = \frac{\Gamma(\kappa,-2c(s+1))}{(s+1)^\kappa\Gamma(\kappa)}$$

and solving for G_c, we have

$$G_c\left(\frac{s}{s+1}\right) = \frac{\Gamma(\kappa,-2c(s+1))}{\sqrt{\Gamma(\kappa)}\sqrt{\Gamma(\kappa,-2c)}}$$

Examining this result, we have assumed $s > 0$, so $s/(s+1)$ is a number in $(0,1)$ and $c < 0$, so $-2c(s+1)$ is a number on the positive real axis. If z satisfies $0 < z < 1$ and $z = s/(s+1)$, then $-2c(1+s) = -2c/(1-z)$ and this result can be rewritten as

$$G_c(z) = \frac{\Gamma(\kappa,-2c/(1-z))}{\sqrt{\Gamma(\kappa)}\sqrt{\Gamma(\kappa,-2c)}} \tag{21}$$

The function G_c is in $H^2(\beta_\kappa)$, so it is analytic in the unit disk. The formulation of the incomplete Gamma function in Equation (16) is analytic in the right half plane, so if z is in the disk then $-2c/(1-z)$ is in the right half plane. Thus, we see that $G_c(z)$ and the expression on the right side of Equation (21) are both analytic for z in the disk and they agree when $0 < z < 1$. This means that they are equal for all z in the disk and Equation (21) gives the extremal function for invariant subspace H_c in the weighted Bergman space $A^2_{\kappa-2} = H^2(\beta_\kappa)$. (Notice that for the choice we have made, $G_c(0) > 0$, so $\zeta = 1$ and our expressions are correct.) □

The dependence of the above formula on the incomplete Gamma function is perhaps disappointing, but it is what it is! However, for κ an integer, the incomplete Gamma function can be explicitly computed by integration by parts. For example, for $\kappa = 1$, the usual Hardy space, and $\kappa = 2$, the usual Bergman space, where the results are already well known, as we would expect, our result agrees with previous results:

For the usual Hardy space, $\kappa = 1$, we have $\Gamma(1) = 1$, and

$$\Gamma(1, w) = \int_w^\infty e^{-t}\, dt = e^{-w}$$

so, in this case,

$$G_c(z) = \frac{\Gamma(1, -2c/(1-z))}{\sqrt{\Gamma(1)}\sqrt{\Gamma(1, -2c)}} = e^{c\left(\frac{1+z}{1-z}\right)}$$

as we would expect.

For the usual Bergman space, $\kappa = 2$, we have $\Gamma(2) = 1$, and

$$\Gamma(2, w) = \int_w^\infty te^{-t}\, dt = (w+1)e^{-w}$$

so, in this case,

$$G_c(z) = \frac{\Gamma(2, -2c/(1-z))}{\sqrt{\Gamma(2)}\sqrt{\Gamma(2, -2c)}} = \frac{1}{(1-2c)^{1/2}}\left(1 - \frac{2c}{1-z}\right) e^{c\frac{1+z}{1-z}}$$

as we would expect.

For $\kappa = 3$, we have $\Gamma(3) = 2$, and

$$\Gamma(3, w) = \int_w^\infty t^2 e^{-t}\, dt = (w^2 + 2w + 2)e^{-w}$$

so, in this case,

$$G_c(z) = \frac{\Gamma(3, -2c/(1-z))}{\sqrt{\Gamma(3)}\sqrt{\Gamma(3, -2c)}} = \frac{1}{(1-2c+2c^2)^{1/2}}\left(1 - \frac{2c}{1-z} + \frac{2c^2}{(1-z)^2}\right) e^{c\frac{1+z}{1-z}}$$

A closed subspace of a reproducing kernel Hilbert space is also a reproducing kernel Hilbert space and it is often of interest to describe the reproducing kernels for the subspace more explicitly than just the 'projections of those for the larger space'. In the following theorem, we give an explicit formula for the reproducing kernels for the atomic inner function subspaces of the weighted Bergman spaces we are considering. Corollary 27 gives these restricted kernel functions for the usual Bergman space ($\kappa = 2$) which had previously been computed in a different way by Yang [29] or see the report in [28].

Theorem 26. *For $0 < r < 1$, let P_r be the orthogonal projection onto the subspace $H_{(\ln r)/2}$ in $H^2(\beta_\kappa)$. If u is any point of the open unit disk, then for $K_u(z) = (1 - \overline{u}z)^{-\kappa}$*

$$(P_r K_u)(z) = \frac{1}{\Gamma(\kappa)(1-\overline{u}z)^\kappa}\Gamma\left(\kappa, -\frac{(\ln r)(1-\overline{u}z)}{(1-\overline{u})(1-z)}\right)$$

Proof. Let us first consider the case $0 \le u < 1$.

Let w belong to the unit disk. The subspace $H_{(\ln r)/2}$ is closed, therefore $K_w = b_w + d_w$ where b_w is in $H_{(\ln r)/2}$ and d_w is in $H^{\perp}_{(\ln r)/2}$. Now if g is in $H_{(\ln r)/2}$, then

$$g(w) = \langle g, K_w \rangle = \langle g, b_w \rangle + \langle g, d_w \rangle = \langle g, b_w \rangle$$

Thus $\langle g, b_w \rangle = g(w)$ for each g in $H_{(\ln r)/2}$ and each w in the disk; that is, b_w is the point evaluation kernel for the Hilbert subspace $H_{(\ln r)/2}$. To compute b_w we put $b_w(z) = b(z, w)$. The subspace $H_{(\ln r)/2}$ is an invariant subspace of the operator W_{f_t,φ_t} which is self-adjoint on A^2 hence the restriction of W_{f_t,φ_t} to $H_{(\ln r)/2}$ is a self-adjoint operator on $H_{(\ln r)/2}$. Moreover, because $H_{(\ln r)/2}$ is invariant for both C_{φ_t} and T_{f_t}, we get

$$W_{f_t,\varphi_t}(b(z,w)) = f_t(z)b(\varphi_t(z), w)$$

and

$$W^*_{f_t,\varphi_t}(b(z,w)) = \overline{f_t(w)}b(z, \varphi_t(w))$$

but $W^*_{f_t,\varphi_t}$ is self-adjoint on $H_{(\ln r)/2}$ therefore

$$\overline{f_t(w)}b(z, \varphi_t(w)) = f_t(z)b(\varphi_t(z), w)$$

hence,

$$b(z, \varphi_t(w)) = \frac{f_t(z)}{\overline{f_t(w)}} b(\varphi_t(z), w) \tag{22}$$

Combining Equation (17), with the known relationship between extremal functions and the kernels, (see, for example, [17, pp. 57-59]), we see that

$$\frac{b(z,0)}{\sqrt{b(0,0)}} = G_c(z) = \frac{\Gamma(\kappa, -(\ln r)/(1-z))}{\sqrt{\Gamma(\kappa)}\sqrt{\Gamma(\kappa, -(\ln r))}}$$

Let $z = 0$ in the above and we get

$$\sqrt{b(0,0)} = \frac{\sqrt{\Gamma(\kappa, -(\ln r))}}{\sqrt{\Gamma(\kappa)}}$$

Therefore

$$b(z,0) = \frac{\Gamma(\kappa, -(\ln r)/(1-z))}{\Gamma(\kappa)}$$

Now by letting $w = 0$ in Equation (22) we get,

$$b(z, \varphi_t(0)) = \frac{(1+t-tz)^{-\kappa}}{(1+t)^{-\kappa}} \frac{\Gamma(\kappa, -(\ln r)/(1-\varphi_t(z)))}{\Gamma(\kappa)}$$

Let $\varphi_t(0) = u$ then $t/(t+1) = u$ hence $t = u/(1-u)$ and $1 + t = 1/(1-u)$. Substituting this value for t in φ_t we get $\varphi_t(z) = \dfrac{\frac{u}{1-u} + (1 - \frac{u}{1-u})z}{1 + \frac{u}{1-u} - \frac{u}{1-u}z}$. This results in

$\varphi_t(z) = \dfrac{u + (1-2u)z}{1-uz}$ and $1 - \varphi_t(z) = \dfrac{1 - u + uz - z}{1-uz} = \dfrac{(1-u)(1-z)}{1-uz}$. Then,

$$\begin{aligned} b(z,u) &= \frac{(\frac{1}{1-u} - \frac{u}{1-u}z)^{-\kappa}}{(\frac{1}{1-u})^{-\kappa}} \frac{\Gamma(\kappa, -(\ln r)/(\frac{(1-u)(1-z)}{1-uz}))}{\Gamma(\kappa)} \\ &= \frac{\Gamma(\kappa, -(\ln r)/(\frac{(1-u)(1-z)}{1-uz}))}{\Gamma(\kappa)(1-uz)^{\kappa}} \end{aligned}$$

To summarize the argument so far, we have shown that for $0 \leq u < 1$, we have

$$(P_r K_u)(z) = \frac{1}{\Gamma(\kappa)(1-uz)^{\kappa}} \Gamma\left(\kappa, -\frac{(\ln r)(1-uz)}{(1-u)(1-z)}\right)$$

Since the function $K_u(z)$ is analytic in z and conjugate analytic in u for z and u in the unit disk, the same is true of $(P_r K_u)(z)$. The function

$$\frac{1}{\Gamma(\kappa)(1-\overline{u}z)^{\kappa}} \Gamma\left(\kappa, -\frac{(\ln r)(1-\overline{u}z)}{(1-\overline{u})(1-z)}\right)$$

in the conclusion above is also analytic in z and conjugate analytic in u for z and u in the unit disk. Since, for each z in the disk, this function agrees with the function $(P_r K_u)(z)$ for $0 \leq u < 1$, they are the same function for all u in the unit disk. In other words, they are the same for all z and u in the unit disk, as we were to prove. □

Finally, we specialize the result of Theorem 26 to get the restricted kernels for the usual Bergman space.

Corollary 27. *For $0 < r < 1$, let P_r be the projection onto the subspace $H_{(\ln r)/2}$ in the Bergman space A^2, which is $H^2(\beta_\kappa)$ for $\kappa = 2$. If u is any point of the open unit disk, then for $K_u(z) = (1-\overline{u}z)^{-2}$*

$$(P_r K_u)(z) = \frac{1}{(1-\overline{u}z)^2} \left(1 - \frac{(\ln r)(1-\overline{u}z)}{(1-\overline{u})(1-z)}\right) e^{\frac{(\ln r)(1-\overline{u}z)}{(1-\overline{u})(1-z)}}$$

Proof. According to Theorem 26 using $\kappa = 2$,

$$(P_r K_u)(z) = \frac{1}{\Gamma(2)(1-\overline{u}z)^2} \Gamma\left(2, -\frac{(\ln r)(1-\overline{u}z)}{(1-\overline{u})(1-z)}\right)$$

Since $\Gamma(2, w) = (w+1)e^{-w}$, we have

$$(P_r K_u)(z) = \frac{1}{(1-\overline{u}z)^2} \left(1 - \frac{(\ln r)(1-\overline{u}z)}{(1-\overline{u})(1-z)}\right) e^{\frac{(\ln r)(1-\overline{u}z)}{(1-\overline{u})(1-z)}}$$

as we were to prove. □

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