HERMITIAN WEIGHTED COMPOSITION OPERATORS ON $\mathbf{H}^2$

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Abstract. Weighted composition operators have been related to products of composition operators and their adjoints and to isometries of Hardy spaces. In this paper, we identify the Hermitian weighted composition operators on H^2 and compute their spectral measures. Some relevant semigroups are studied. The resulting ideas can be used to find the polar decomposition, the absolute value, and the Aluthge transform of some composition operators on H^2.

1. Introduction

If f is in H^∞ and φ is analytic map of the unit disk into itself, the *weighted composition operator on H^2 with symbols f and φ* is the operator $W_{f,\varphi} = T_f C_\varphi$, where T_f is the analytic Toeplitz operator given by $T_f(h) = fh$ for h in H^2, C_φ is the composition operator on H^2 given by $C_\varphi(h) = h \circ \varphi$. Clearly, if f is bounded on the disk, then $W_{f,\varphi}$ is bounded on H^2 and

$$\|W_{f,\varphi}\| = \|T_f C_\varphi\| \leq \|f\|_\infty \|C_\varphi\|.$$

Although it will have little impact on our work, it is not necessary for f to be bounded for $W_{f,\varphi}$ to be bounded (see [10]).

Weighted composition operators have been studied occasionally over the past few decades, but have usually arisen in answering other questions related to operators on spaces of analytic functions, such as questions about multiplication operators or composition operators. For example, Forelli [9] showed that the only isometries of H^p for $p \neq 2$ are weighted composition operators and that the isometries for H^p with $p \neq 2$ have analogues that are isometries of H^2 (but there are also many other isometries of H^2). Weighted composition operators also arise in the description of commutants of analytic Toeplitz operators (see for example [2, 3]) and in the adjoints of composition operators (see for example [5, 8, 6]). Only recently have investigators begun to study the properties of weighted composition operators in general (see [10]).

In this paper, we examine the question "Which weighted composition operators on H^2 are self-adjoint?" For composition operators on H^2, that is, the case where the weight function, f, is identically one, the situation is trivial: the only self-adjoint composition operators have symbol $\varphi(z) = rz$ with $-1 \leq r \leq 1$. For

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weighted composition operators, the situation is more interesting, but we will see that self-adjoint weighted composition operators are rare and there is a strong connection between the symbols for composition and multiplication. We first show that for all Hermitian weighted composition operators on H^2, both symbols that define the operators are linear fractional maps. Then, we show that the possible symbols are divided into three cases that yield compact operators, real multiples of self-adjoint unitary operators, and operators with continuous spectrum. In each of these cases, we compute the spectral measures for the operators. In addition, we include some results about semigroups of weighted composition operators. Finally, we use the results of these analyses to find the polar decomposition, the absolute value, and the Aluthge transform of some composition operators on H^2.

We begin by proving a result that allows us to choose standard forms for the operators under study.

Proposition 1. *Let f be in H^∞ and let φ be an analytic map of the unit disk into itself. For θ a real number, let U_θ be the unitary composition operator given by $(U_\theta h)(z) = h(e^{i\theta}z)$ for h in H^2. Then*

$$U_\theta^* T_f C_\varphi U_\theta = T_{\tilde{f}} C_{\widetilde{\varphi}}$$

where $\tilde{f}(z) = f(e^{-i\theta}z)$ and $\widetilde{\varphi}(z) = e^{i\theta}\varphi(e^{-i\theta}z)$

Proof. Let $\tilde{f}(z) = f(e^{-i\theta}z)$ and $\widetilde{\varphi}(z) = e^{i\theta}\varphi(e^{-i\theta}z)$ as in the conclusion of the statement above. For h in H^2,

$$\begin{aligned}(U_\theta^* T_f C_\varphi U_\theta h)(z) &= (U_\theta^* T_f C_\varphi)(h(e^{i\theta}z)) = (U_\theta^*)(f(z)h(e^{i\theta}\varphi(z))) \\ &= f(e^{-i\theta}z)h(e^{i\theta}\varphi(e^{-i\theta}z)) = \tilde{f}(z)h(\widetilde{\varphi}(z)) \\ &= (T_{\tilde{f}} C_{\widetilde{\varphi}} h)(z)\end{aligned}$$

Since this is true for every h in H^2, the conclusion follows. □

This proposition will permit us to choose convenient symbols for the weighted composition operators we study without losing any generality of the properties of the operators we are trying to understand.

Corollary 2. *For f in H^∞ and φ an analytic map of the unit disk into itself, there are g in H^∞ and ψ an analytic map of the unit disk into itself with $\psi(0) \geq 0$ so that the weighted composition operator $W_{g,\psi}$ is unitarily equivalent to $W_{f,\varphi}$.*

Proof. Choose θ in Proposition 1 so that $\widetilde{\varphi}(0) = e^{i\theta}\varphi(e^{-i\theta}0) = e^{i\theta}\varphi(0)$ is non-negative. Letting $g = \tilde{f}$ and $\psi = \widetilde{\varphi}$ satisfies the conclusion of the Corollary. □

The following observation can be helpful in considering the action of weighted composition operators on products of functions, a situation that arises frequently.

Proposition 3. *Let f be in H^∞ and let φ be an analytic map of the unit disk into itself. If g and h are functions in H^2 such that gh is also in H^2, then $(W_{f,\varphi})(gh) = (C_\varphi h)(W_{f,\varphi}g) = (C_\varphi g)(W_{f,\varphi}h)$.*

Proof.

$$\begin{aligned}(W_{f,\varphi}gh)(z) &= (T_f C_\varphi gh)(z) = f(z)g(\varphi(z)h(\varphi(z)) \\ &= (T_f C_\varphi g)(z)(C_\varphi h)(z) = (C_\varphi h)(z)(W_{f,\varphi}g)(z)\end{aligned}$$

Similarly,

$$\begin{aligned}(W_{f,\varphi}gh)(z) &= (T_f C_\varphi gh)(z) = f(z)g(\varphi(z)h(\varphi(z)) = g(\varphi(z)f(z)h(\varphi(z)) \\ &= (C_\varphi g)(z)(T_f C_\varphi h)(z) = (C_\varphi g)(z)(W_{f,\varphi}h)(z)\end{aligned}$$

It is not difficult to see that all the relevant products make sense in H^2, so the conclusion follows. □

The following lemma isolates a calculation that will be useful in the coming sections.

Lemma 4. *For a_1 real and a_0 a complex number, let $\varphi(z) = a_0 + a_1 z/(1 - \overline{a_0}z)$. Let b be a fixed point of φ and let $\psi(z) = (z - b)/(\overline{b}z - 1)$.*
Then 1) $b = 0$ when $a_0 = 0$ and

$$\begin{aligned}b &= \frac{1 + |a_0|^2 - a_1 \pm \sqrt{(1 + |a_0|^2 - a_1)^2 - 4|a_0|^2}}{2\overline{a_0}} \\ &= \frac{2a_0}{1 + |a_0|^2 - a_1 \mp \sqrt{(1 + |a_0|^2 - a_1)^2 - 4|a_0|^2}} \qquad \text{for } a_0 \neq 0\end{aligned}$$

and 2) $\psi(\varphi(z)) = \alpha\psi(z)$ where

$$\alpha = \frac{a_1 - |a_0|^2 + b\overline{a_0}}{1 - \overline{b}a_0}$$

and, when $a_1 \leq (1 - |a_0|)^2$, then $b\overline{a_0} = \overline{b}a_0$ which implies α is real and $\alpha = \varphi'(b)$.

Proof. Rewriting φ, we see that

$$\varphi(z) = a_0 + \frac{a_1 z}{1 - \overline{a_0}z} = \frac{(a_1 - |a_0|^2)z + a_0}{1 - \overline{a_0}z}$$

Then,

$$\begin{aligned}\psi(\varphi(z)) &= \frac{\frac{(a_1 - |a_0|^2)z + a_0}{1 - \overline{a_0}z} - b}{\overline{b}\frac{(a_1 - |a_0|^2)z + a_0}{1 - \overline{a_0}z} - 1} \\ &= \frac{(a_1 - |a_0|^2)z + a_0 - b + b\overline{a_0}z}{\overline{b}(a_1 - |a_0|^2)z + \overline{b}a_0 - 1 + \overline{a_0}z} \\ &= \frac{(a_1 - |a_0|^2 + b\overline{a_0})z + a_0 - b}{(\overline{b}(a_1 - |a_0|^2) + \overline{a_0})z + \overline{b}a_0 - 1} \\ &= \frac{1}{1 - \overline{b}a_0}\frac{(a_1 - |a_0|^2 + b\overline{a_0})z + a_0 - b}{\frac{(a_1 - |a_0|^2)\overline{b} + \overline{a_0}}{1 - a_0\overline{b}}z - 1} \\ &= \frac{1}{1 - \overline{b}a_0}\frac{(a_1 - |a_0|^2 + b\overline{a_0})z + a_0 - b}{\overline{\varphi(b)}z - 1} \\ &= \frac{1}{1 - \overline{b}a_0}\frac{(a_1 - |a_0|^2 + b\overline{a_0})z + a_0 - b}{\overline{b}z - 1}\end{aligned}$$

Since $\psi(b) = 0$ and $\varphi(b) = b$, we see that $\psi(\varphi(b)) = 0$ as well, so in particular, $z - b$ divides the numerator of the expression above for $\psi(\varphi(z))$. More explicitly, the statement that $\varphi(b) = b$ means that

$$\frac{(a_1 - |a_0|^2)b + a_0}{1 - \overline{a_0}b} = b$$

or that

$$-(a_1 - |a_0|^2)b - \overline{a_0}b^2 = a_0 - b \tag{1}$$

Now,

$$\begin{aligned} (a_1 - |a_0|^2 + b\overline{a_0})(z - b) &= (a_1 - |a_0|^2 + b\overline{a_0})z - (a_1 - |a_0|^2)b - \overline{a_0}b^2 \\ &= (a_1 - |a_0|^2 + b\overline{a_0})z + a_0 - b \end{aligned}$$

Using this equality in the calculation above, we see that

$$\psi(\varphi(z)) = \frac{a_1 - |a_0|^2 + b\overline{a_0}}{1 - \bar{b}a_0} \frac{z - b}{\bar{b}z - 1} = \alpha\psi(z)$$

Rewriting Equation (1), we have b is a fixed point of φ if and only if

$$\overline{a_0}b^2 - (1 + |a_0|^2 - a_1)b + a_0 = 0 \tag{2}$$

If $a_0 = 0$, we see $b = 0$ is a fixed point of φ. If $a_0 \neq 0$, the quadratic equation gives the fixed points as

$$b = \frac{1 + |a_0|^2 - a_1 \pm \sqrt{(1 + |a_0|^2 - a_1)^2 - 4|a_0|^2}}{2\overline{a_0}}$$

rationalizing the numerator, we get

$$b = \frac{2a_0}{1 + |a_0|^2 - a_1 \mp \sqrt{(1 + |a_0|^2 - a_1)^2 - 4|a_0|^2}}$$

Notice that if $a_1 \leq (1 - |a_0|)^2$, then $1 + |a_0|^2 - a_1 \geq 2|a_0|$ which means $\sqrt{(1 + |a_0|^2 - a_1)^2 - 4|a_0|^2}$ is a real number. It follows that $\overline{a_0}b = \bar{b}a_0$ are both real as well. Thus, in this case,

$$\alpha = \frac{a_1 - |a_0|^2 + b\overline{a_0}}{1 - \overline{a_0}b}$$

Now, by Equation (2),

$$\begin{aligned} (a_1 - |a_0|^2 + b\overline{a_0})(1 - \overline{a_0}b) &= a_1 - |a_0|^2 + b\overline{a_0} - a_1\overline{a_0}b + |a_0|^2\overline{a_0}b - b^2\overline{a_0}^2 \\ &= a_1 - \overline{a_0}\left(\overline{a_0}b^2 - (1 + |a_0|^2 - a_1)b + a_0\right) \\ &= a_1 - \overline{a_0}(0) = a_1 \end{aligned}$$

This means that

$$\alpha = \frac{a_1 - |a_0|^2 + b\overline{a_0}}{1 - \overline{a_0}b} = \frac{(a_1 - |a_0|^2 + b\overline{a_0})(1 - \overline{a_0}b)}{(1 - \overline{a_0}b)^2} = \frac{a_1}{(1 - \overline{a_0}b)^2} = \varphi'(b)$$

□

2. Hermitian weighted composition operators

We first investigate which combinations of weights f and maps of the disk φ give rise to Hermitian weighted composition operators. Not surprisingly, self-adjointness significantly restricts the possible symbols for the weighted composition operators.

Theorem 5. *Let f be in H^∞ and let φ be an analytic map of the unit disk into itself. If the weighted composition operator $W_{f,\varphi} = T_f C_\varphi$ is Hermitian on H^2, then $f(0)$ and $\varphi'(0)$ are real and $\varphi(z) = a_0 + a_1 z/(1-\overline{a_0}z)$ and $f(z) = c/(1-\overline{a_0}z)$ where $a_0 = \varphi(0)$, $a_1 = \varphi'(0)$, and $c = f(0)$.*

Conversely, let a_0 be in $\mathbb{D}$, and let c and a_1 be real numbers. If $\varphi(z) = a_0 + a_1 z/(1-\overline{a_0}z)$ maps the unit disk into itself and $f(z) = c/(1-\overline{a_0}z)$, then the weighted composition operator $W_{f,\varphi} = T_f C_\varphi$ is Hermitian.

Proof. Suppose $T_f C_\varphi$ is a weighted composition operator on H^2. For α in the open unit disk $\mathbb{D}$, let K_α be the evaluation kernel for H^2, that is,

$$K_\alpha(z) = \frac{1}{1-\overline{\alpha}z}$$

Then

$$\left(T_f C_\varphi K_\alpha\right)(z) = \left(T_f C_\varphi\right)\left(\frac{1}{1-\overline{\alpha}z}\right) = T_f\left(\frac{1}{1-\overline{\alpha}\varphi(z)}\right) = \frac{f(z)}{1-\overline{\alpha}\varphi(z)}$$

On the other hand,

$$\begin{aligned}\left(T_f C_\varphi\right)^*\left(K_\alpha\right)(z) &= \left(C_\varphi^* T_f^*\right)\left(K_\alpha\right)(z) = \overline{f(\alpha)}C_\varphi^*\left(K_\alpha\right)(z) \\ &= \overline{f(\alpha)}K_{\varphi(\alpha)}(z) = \frac{\overline{f(\alpha)}}{1-\overline{\varphi(\alpha)}z}\end{aligned}$$

Thus, $T_f C_\varphi$ is Hermitian if and only if

$$\frac{f(z)}{1-\overline{\alpha}\varphi(z)} = \frac{\overline{f(\alpha)}}{1-\overline{\varphi(\alpha)}z} \tag{3}$$

for all α and z in the unit disk.

In particular, letting $\alpha = 0$ in Equation (3), we get

$$f(z) = \frac{\overline{f(0)}}{1-\overline{\varphi(0)}z}$$

for all z in the disk. Setting $z = 0$, we get $f(0) = \overline{f(0)}$, so that $f(0)$ is real. Defining c and a_0 by $c = f(0)$ and $a_0 = \varphi(0)$, we can write f as

$$f(z) = \frac{c}{1-\overline{a_0}z} \tag{4}$$

Combining Equations (3) and (4) we get

$$\frac{1-\overline{\alpha}\varphi(z)}{f(z)} = \frac{(1-\overline{\alpha}\varphi(z))(1-\overline{a_0}z)}{c}$$

and

$$\frac{1-\overline{\varphi(\alpha)}z}{\overline{f(\alpha)}} = \frac{(1-\overline{\varphi(\alpha)}z)(1-\overline{\alpha}a_0)}{c}$$

which means $T_f C_\varphi$ is Hermitian if and only if

$$(1 - \overline{\alpha}\varphi(z))(1 - \overline{a_0}z) = (1 - \overline{\varphi(\alpha)}z)(1 - \overline{\alpha}a_0)$$

for all α and z in the unit disk.

Notice that the expression on the right is a polynomial of degree one in the variable z. This means that the expression on the left

$$(1 - \overline{\alpha}\varphi(z))(1 - \overline{a_0}z) = 1 - \overline{\alpha}\varphi(z) - \overline{a_0}z + \overline{\alpha a_0}z\varphi(z) \tag{5}$$

must also be a polynomial of degree one in z.

Suppose

$$\varphi(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots$$

is the Taylor expansion for φ. Then replacing $\varphi(z)$ with this series in Equation (5), we see that

$$-\overline{\alpha}a_j z^j + \overline{\alpha a_0}z a_{j-1} z^{j-1} = 0$$

for each integer j for which $j \geq 2$. In other words, $a_j = \overline{a_0}a_{j-1}$ for $j \geq 2$. In particular, $a_2 = \overline{a_0}a_1$, which means $a_3 = \overline{a_0}a_2 = \overline{a_0}^2 a_1$, and continuing, we get $a_j = \overline{a_0}^{j-1}a_1$ for $j \geq 2$.

Substituting into the Taylor series, we see that

$$\varphi(z) = a_0 + a_1 z + \overline{a_0}a_1 z^2 + \overline{a_0}^2 a_1 z^3 + \overline{a_0}^3 a_1 z^4 + \cdots = a_0 + \frac{a_1 z}{1 - \overline{a_0}z} \tag{6}$$

Using the expressions for f and φ from Equations (4) and (6) in Equation (3), we get

$$\frac{\frac{c}{1-\overline{a_0}z}}{1 - \overline{\alpha}(a_0 + \frac{a_1 z}{1-\overline{a_0}z})} = \frac{\frac{c}{1-a_0\overline{\alpha}}}{1 - (\overline{a_0} + \frac{\overline{a_1 \alpha}}{1-a_0\overline{\alpha}})z}$$

or

$$\frac{\frac{c}{1-\overline{a_0}z}}{1 - \overline{\alpha}a_0 - \frac{\overline{\alpha}a_1 z}{1-\overline{a_0}z}} = \frac{\frac{c}{1-a_0\overline{\alpha}}}{1 - \overline{a_0}z - \frac{\overline{a_1 \alpha}z}{1-a_0\overline{\alpha}}}$$

Clearing the fractions on both sides of this expression, we see that this implies

$$(1 - \overline{a_0}z)(1 - a_0\overline{\alpha}) - a_1\overline{\alpha}z = (1 - \overline{a_0}z)(1 - a_0\overline{\alpha}) - \overline{a_1}\,\overline{\alpha}z$$

for all α and z in the disk. In particular, this means $a_1 = \overline{a_1}$, so that $a_1 = \varphi'(0)$ is real.

Conversely, if a_1, a real number, and a_0, in the unit disk, are such that

$$\varphi(z) = a_0 + \frac{a_1 z}{(1 - \overline{a_0}z)}$$

maps the disk into itself and $f(z) = c/(1 - \overline{a_0}z)$ for c a real number, then a straightforward computation shows Equation (3) holds for all α and z in the disk, which means that $T_f C_\varphi$ is Hermitian. □

Theorem 5 assumes that the function φ maps the disk into itself. Of course, not all combinations of the parameters a_0 and a_1 yield a mapping of the disk into itself. We next consider which combinations of these parameters give a mapping of the disk into the disk. Corollary 2 shows that, without loss of generality, we may assume $a_0 = \varphi(0)$ is real and non-negative. The following easy calculation gives the conditions on real numbers a_0 and a_1 that produce maps of the disk into itself.

Lemma 6. *Let* a_0 *be real and non-negative and and let* a_1 *be real. Then* $\varphi(z) = a_0 + a_1 z/(1 - a_0 z)$ *maps the open unit disk into itself if and only if*

$$0 \le a_0 < 1 \quad \textit{and} \quad -1 + a_0^2 \le a_1 \le (1 - a_0)^2 \tag{7}$$

Proof. Suppose a_0 and a_1 are real with $a_0 \ge 0$ and $\varphi(z) = a_0 + a_1 z/(1 - a_0 z)$. Since φ is a linear fractional map with real coefficients, Theorem 10 of [8] says that φ maps the open unit disk into itself if and only if φ maps the open interval $(-1, 1)$ into itself. In particular, if φ maps the open disk into itself, then $a_0 = \varphi(0)$ must lie in the open unit disk, that is, $0 \le a_0 < 1$. For such a_0, the map φ is continuous on the closed interval $[-1, 1]$ and

$$\varphi'(z) = \frac{a_1}{(1 - a_0 z)^2} \ne 0$$

so φ is either increasing on $(-1, 1)$ or decreasing on $(-1, 1)$ depending on the sign of a_1. Thus, for $0 \le a_0 < 1$, φ maps $(-1, 1)$ into itself, hence the unit disk into itself, if and only if $-1 \le \varphi(-1) \le 1$ and $-1 \le \varphi(1) \le 1$.

Since

$$\varphi(-1) = a_0 - \frac{a_1}{1 + a_0} = \frac{a_0 + a_0^2 - a_1}{1 + a_0}$$

$-1 \le \varphi(-1) \le 1$ if and only if

$$-1 - a_0 \le a_0 + a_0^2 - a_1 \le 1 + a_0$$

or

$$-1 - a_0 - a_0 - a_0^2 \le -a_1 \le 1 + a_0 - a_0 - a_0^2$$

That is, $-1 \le \varphi(-1) \le 1$ if and only if

$$-1 + a_0^2 \le a_1 \le (1 + a_0)^2$$

Since

$$\varphi(1) = a_0 + \frac{a_1}{1 - a_0} = \frac{a_0 - a_0^2 + a_1}{1 - a_0}$$

$-1 \le \varphi(1) \le 1$ if and only if

$$-1 + a_0 \le a_0 - a_0^2 + a_1 \le 1 - a_0$$

or

$$-1 + a_0 - a_0 + a_0^2 \le a_1 \le 1 - a_0 - a_0 + a_0^2$$

That is, $-1 \le \varphi(1) \le 1$ if and only if

$$-1 + a_0^2 \le a_1 \le (1 - a_0)^2$$

Since $0 \le a_0 < 1$, we have $(1 - a_0)^2 \le (1 + a_0)^2$, and we conclude φ maps the unit disk into itself if and only if $-1 + a_0^2 \le a_1 \le (1 - a_0)^2$. □

Corollary 7. *Let* a_1 *be real. Then* $\varphi(z) = a_0 + a_1 z/(1 - \overline{a_0} z)$ *maps the open unit disk into itself if and only if*

$$|a_0| < 1 \quad \textit{and} \quad -1 + |a_0|^2 \le a_1 \le (1 - |a_0|)^2 \tag{8}$$

Proof. The function φ maps the open disk into itself if and only if the function $\widetilde{\varphi}$ given by $\widetilde{\varphi}(z) = e^{i\theta}\varphi(e^{-i\theta} z)$ maps the open unit disk into itself. Choosing θ so that $\widetilde{\varphi}$ satisfies the hypotheses of Lemma 6 gives $\widetilde{\varphi}(z) = |a_0| + a_1 z/(1 - |a_0| z)$. The corollary now follows immediately from Lemma 6. □

In the next three sections, we explore the three cases $a_1 = -1+|a_0|^2$, $-1+|a_0|^2 < a_1 < (1-|a_0|)^2$, and $a_1 = (1-|a_0|)^2$ which are quite different from each other.

When $0 \le a_0 < 1$, the first case, $a_1 = -1 + a_0^2$, gives

$$\varphi(-1) = a_0 - \frac{a_1}{1+a_0} = a_0 + \frac{1-a_0^2}{1+a_0} = a_0 + 1 - a_0 = 1$$

and

$$\varphi(1) = a_0 + \frac{a_1}{1-a_0} = a_0 + \frac{-1+a_0^2}{1-a_0} = a_0 - 1 - a_0 = -1$$

In this case, φ is an automorphism of the disk. We will see (Theorem 8 and the comment following) that in this case, $W_{f,\varphi}$ is a multiple of a Hermitian isometric weighted composition operator.

When $0 \le a_0 < 1$, the second case, $-1 + a_0^2 < a_1 < (1-a_0)^2$, gives

$$a_0 - \frac{(1-a_0)^2}{1+a_0} < a_0 - \frac{a_1}{1+a_0} < a_0 + \frac{1-a_0^2}{1+a_0}$$

so

$$\frac{a_0 + a_0^2 - 1 + 2a_0 - a_0^2}{1+a_0} < \varphi(-1) < a_0 + 1 - a_0$$

or

$$-1 \le -1 + \frac{4a_0}{1+a_0} < \varphi(-1) < 1$$

That is, $-1 < \varphi(-1) < 1$. Moreover,

$$a_0 + \frac{-1+a_0^2}{1-a_0} < a_0 + \frac{a_1}{1-a_0} < a_0 + \frac{(1-a_0)^2}{1-a_0}$$

so

$$-1 \le a_0 - 1 + a_0 < \varphi(1) < a_0 + 1 - a_0 = 1$$

That is, $-1 < \varphi(1) < 1$. Since φ is a linear fractional map with real coefficients and maps the unit disk into itself, conformality at ± 1 means that φ maps the closed unit disk onto a closed disk whose diameter is the closed interval with end points $\varphi(-1)$ and $\varphi(1)$. Thus, in this case, φ maps the closed unit disk into the open unit disk and C_φ and (therefore) $W_{f,\varphi}$ are compact. The analysis is based on identifying the eigenvectors and eigenvalues of the operator as in the first case (Theorem 9).

When $0 < a_0 < 1$, the third case, $a_1 = (1-a_0)^2$, gives

$$\varphi(-1) = a_0 - \frac{a_1}{1+a_0} = a_0 - \frac{(1-a_0)^2}{1+a_0} = \frac{a_0 + a_0^2 - 1 + 2a_0 - a_0^2}{1+a_0} = -1 + \frac{4a_0}{1+a_0}$$

so $-1 < \varphi(-1) < 1$. Moreover,

$$\varphi(1) = a_0 + \frac{a_1}{1-a_0} = a_0 + \frac{(1-a_0)^2}{1-a_0} = a_0 + 1 - a_0 = 1$$

In this case, φ is not an automorphism of the disk, but $\varphi(1) = 1$. (The case $a_0 = 0$ and $a_1 = (1-a_0)^2 = 1$ is $\varphi(z) = z$ and the resulting weighted composition operators $T_f C_\varphi$ are constant multiples of the identity, an uninteresting case.) We will see (Section 5) that in this case, each $W_{f,\varphi}$ is a member of a continuous semigroup of Hermitian operators so the theory of semigroups can help us study the structure of these operators.

Because Corollary 2 shows that every Hermitian weighted composition operator is unitarily equivalent to one with $0 \le a_0 < 1$ and Proposition 1 gives the explicit unitary equivalence, we see that the three cases above persist. The first case, $a_1 = -1 + |a_0|^2$, corresponds to φ being an automorphism of the unit disk and $W_{f,\varphi}$ is a multiple of a Hermitian isometric weighted composition operator. In the second case, $-1 + |a_0|^2 < a_1 < (1 - |a_0|)^2$, the map φ takes the closed unit disk into the open disk and $W_{f,\varphi}$ is compact. In the third case, $a_1 = (1 - |a_0|)^2$ where $0 < |a_0| < 1$, the map φ, not an automorphism of the disk, has fixed point $b = |a_0|/\overline{a_0} = a_0/|a_0|$ on the unit circle and $\varphi'(b) = 1$. For this case, we will see that each $W_{f,\varphi}$ is a member of a continuous semigroup that will allow us to analyze the operators.

3. Hermitian isometric weighted composition operators

We consider first the case $|a_0| < 1$ and $a_1 = -1 + |a_0|^2$ for which

$$\varphi(z) = a_0 + \frac{(-1 + |a_0|^2)z}{1 - \overline{a_0}z} = \frac{z - a_0}{\overline{a_0}z - 1}$$

is an automorphism of the disk onto itself. By Theorem 5, $T_f C_\varphi$ will be a Hermitian weighted composition operator if and only if $f(z) = c/(1 - \overline{a_0}z)$ for some real number c.

We begin doing some computations with $c = 1$. Thus, we consider $T_f C_\varphi$ with $f(z) = (1 - \overline{a_0}z)^{-1}$ and $\varphi(z) = (z - a_0)/(\overline{a_0}z - 1)$.

Motivated by the observation that the map φ satisfies $\varphi(\varphi(z)) = z$,

$$\varphi(\varphi(z)) = \frac{\frac{z-a_0}{\overline{a_0}z-1} - a_0}{\overline{a_0}\frac{z-a_0}{\overline{a_0}z-1} - 1} = \frac{z - a_0 - |a_0|^2 z + a_0}{\overline{a_0}z - |a_0|^2 - \overline{a_0}z + 1} = \frac{(1 - |a_0|^2)z}{1 - |a_0|^2} = z$$

we compute $(T_f C_\varphi)^2$. For h in H^2, we have

$$(T_f C_\varphi T_f C_\varphi h) = (T_f C_\varphi)(f(h \circ \varphi)) = f(f \circ \varphi)(h \circ \varphi \circ \varphi) = f(f \circ \varphi)h$$

The multiplier here is

$$f(z)f(\varphi(z)) = \frac{1}{1 - \overline{a_0}z}\frac{1}{1 - \overline{a_0}\frac{z-a_0}{\overline{a_0}z-1}} = \frac{1}{1 - \overline{a_0}z + \overline{a_0}z - |a_0|^2} = \frac{1}{1 - |a_0|^2}$$

In other words, $(T_f C_\varphi)^2 = (1 - |a_0|^2)^{-1} I$.

From this calculation, we see that it would be better to take $|c| = \sqrt{1 - |a_0|^2}$ so that $f(z) = \pm\sqrt{1 - |a_0|^2}/(1 - \overline{a_0}z)$ and the Hermitian weighted composition operator $T_f C_\varphi$ satisfies $(T_f C_\varphi)^2 = I$, that is, it is also an isometry. Forelli's paper [9] on the isometries of H^p, of course, includes this example.

When $a_0 = 0$, we get the trivial cases $T_f C_\varphi = I$ and $T_f C_\varphi = -I$, but for $|a_0| > 0$, the operators are not so trivial. Self-adjoint isometries have the eigenvalues 1 and -1; we seek the eigenspaces corresponding to these eigenvalues.

For b in the open unit disk and for $j = 0, 1, 2, \cdots$, let

$$e_j(z) = \frac{\sqrt{1 - |b|^2}}{1 - \bar{b}z}\left(\frac{z - b}{\bar{b}z - 1}\right)^j$$

Because the second factor in the expression for e_j is a finite Blaschke product and the first factor is a multiple of the kernel function for evaluation of H^2 functions at b, it is not hard to show that the set $\{e_j : j = 0, 1, 2, \cdots\}$ is an orthonormal set. Further, it is not difficult to see that a function h that is orthogonal to every e_j must vanish with all its derivatives at b. Therefore, the only function in H^2 orthogonal to all the e_j is the zero function and the set $\{e_j\}$ is an orthonormal basis for H^2.

It will be convenient to take b to be a fixed point of φ that is in the open unit disk. Lemma 4 says that if $a_0 \neq 0$ then the fixed points are

$$\begin{aligned} b &= \frac{2a_0}{1 + |a_0|^2 - a_1 \mp \sqrt{(1 + |a_0|^2 - a_1)^2 - 4|a_0|^2}} \\ &= \frac{2a_0}{2 \mp \sqrt{4 - 4|a_0|^2}} = \frac{a_0}{1 \mp \sqrt{1 - |a_0|^2}} \end{aligned}$$

where the second equality comes from our assumption that $a_1 = -1 + |a_0|^2$. It is not difficult to see that the fixed point in the open unit disk is

$$b = \frac{a_0}{1 + \sqrt{1 - |a_0|^2}} = \frac{1 - \sqrt{1 - |a_0|^2}}{\overline{a_0}}$$

Notice that, for this b, in the notation of Lemma 4,

$$e_j = \frac{\sqrt{1 - |b|^2}}{1 - \bar{b}z} \psi(z)^j$$

Lemma 4 says that $\psi(\varphi(z)) = \alpha\psi(z)$ where

$$\alpha = \frac{a_1 - |a_0|^2 + b\overline{a_0}}{1 - \bar{b}a_0}$$

Since we have assumed $0 < |a_0| < 1$ and $a_1 = -1 + |a_0|^2$ so that $a_1 < (1 - |a_0|)^2$, it follows from Lemma 4 that

$$\begin{aligned} \alpha &= \frac{-1 + |a_0|^2 - |a_0|^2 + b\overline{a_0}}{1 - \bar{b}a_0} = \frac{-1 + b\overline{a_0}}{1 - \bar{b}a_0} \\ &= \frac{-1 + b\overline{a_0}}{1 - b\overline{a_0}} = -1 \end{aligned}$$

Calculating, we find

$$\begin{aligned}(T_fC_\varphi e_j)(z) &= \frac{c}{1-\overline{a_0}z}\frac{\sqrt{1-|b|^2}}{1-\bar{b}\frac{z-a_0}{\overline{a_0}z-1}}(\psi(\varphi(z)))^j = \frac{c\sqrt{1-|b|^2}}{1-\overline{a_0}z+\bar{b}z-\bar{b}a_0}(-1)^j\psi(z)^j \\ &= (-1)^j\frac{c\sqrt{1-|b|^2}}{(1-\bar{b}a_0)+(\bar{b}-\overline{a_0})z}\psi(z)^j = (-1)^j\overline{\frac{c}{1-\overline{a_0}b}}\frac{\sqrt{1-|b|^2}}{1-\overline{\left(\frac{b-a_0}{\overline{a_0}b-1}\right)}z}\psi(z)^j \\ &= (-1)^j\overline{f(b)}\frac{\sqrt{1-|b|^2}}{1-\overline{\varphi(b)}z}\psi(z)^j = (-1)^j\overline{f(b)}\frac{\sqrt{1-|b|^2}}{1-\bar{b}z}\psi(z)^j \\ &= (-1)^j\overline{f(b)}e_j(z)\end{aligned}$$

In other words, e_j is an eigenvector for T_fC_φ with eigenvalue $(-1)^j\overline{f(b)}$. Since the set $\{e_j\}$ is an orthonormal basis for H^2, we have determined the spectral measure for the operator T_fC_φ in this case.

For $f(z) = \sqrt{1-|a_0|^2}/(1-\overline{a_0}z)$, we find

$$f(b) = \frac{\sqrt{1-|a_0|^2}}{1-\overline{a_0}\frac{1-\sqrt{1-|a_0|^2}}{\overline{a_0}}} = \frac{\sqrt{1-|a_0|^2}}{1-1+\sqrt{1-|a_0|^2}} = 1$$

Similarly, if we take $f(z) = -\sqrt{1-|a_0|^2}/(1-\overline{a_0}z)$, then $f(b) = -1$. For these choices of the weight function f, the eigenvalues of T_fC_φ are ±1.

The following result formalizes the work of this section.

Theorem 8. *Let a_0 be a point of the open unit disk, $a_0 \neq 0$. For $f(z) = \sqrt{1-|a_0|^2}/(1-\overline{a_0}z)$ and $\varphi(z) = (z-a_0)/(\overline{a_0}z-1)$, the weighted composition operator T_fC_φ is a Hermitian isometry on H^2 with spectrum $\{-1,1\}$.*

Moreover, if b is the fixed point of φ in the open unit disk, and

$$e_j(z) = \frac{\sqrt{1-|b|^2}}{1-\bar{b}z}\left(\frac{z-b}{\bar{b}z-1}\right)^j$$

then the set $\{e_j : j = 0,1,2,\cdots\}$ is an orthonormal basis for H^2 consisting of eigenvectors for T_fC_φ. The eigenspace corresponding to the eigenvalue 1 for T_fC_φ is M_e, the closed span of $\{e_j : j = 0,2,4,\cdots\}$, and the eigenspace corresponding to the eigenvalue -1 for T_fC_φ is M_o, the closed span of $\{e_j : j = 1,3,5,\cdots\}$.

For other choices of weight function for the case $a_1 = -1+|a_0|^2$, that is, real multiples of the function f of Theorem 8, the corresponding weighted composition operator is a multiple of the above operator. Therefore, the eigenspaces are the same and the spectrum is $\{-r,r\}$ for the appropriate real number r.

4. Compact Hermitian weighted composition operators

Next, we consider the case $|a_0| < 1$ and $-1+|a_0|^2 < a_1 < (1-|a_0|)^2$ for which

$$\varphi(z) = a_0 + \frac{a_1 z}{1-\overline{a_0}z}$$

By Theorem 5, $W_{f,\varphi} = T_f C_\varphi$ will be a Hermitian weighted composition operator if and only if $f(z) = c/(1 - \overline{a_0}z)$ for some real number c. The comments at the end of Section 2 explain why, in this case, the operators C_φ and $W_{f,\varphi}$ are compact on H^2. The compactness of C_φ implies that φ has a fixed point in the open unit disk (see, for example, [7, p. 265]).

If b is the fixed point of φ in the open unit disk, for $j = 0, 1, 2, \cdots$, let

$$e_j(z) = \frac{\sqrt{1-|b|^2}}{1-\bar{b}z}\left(\frac{z-b}{\bar{b}z-1}\right)^j$$

be the orthonormal basis in Theorem 8. Then

$$\begin{aligned} T_f C_\varphi e_0 &= C_\varphi^* T_f^* e_0 = \overline{f(b)}\sqrt{1-|b|^2}C_\varphi^* K_b \\ &= \overline{f(b)}\sqrt{1-|b|^2}K_{\varphi(b)} = \overline{f(b)}\sqrt{1-|b|^2}K_b \\ &= \overline{f(b)}e_0 \end{aligned}$$

Now, let's compute the general case using Proposition 3, Lemma 4, the assumption that $a_1 < (1 - |a_0|)^2$, and the fact that, in the notation of Lemma 4, $e_j = e_0\psi^j$. For $j = 1, 2, \cdots$

$$\begin{aligned} T_f C_\varphi e_j &= T_f C_\varphi e_0 \psi^j = T_f C_\varphi(e_0) C_\varphi \psi^j \\ &= (\overline{f(b)}e_0)(\psi \circ \varphi)^j = \overline{f(b)}\alpha^j(e_0\psi^j) \\ &= \overline{f(b)}\alpha^j e_j = \overline{f(b)}\varphi'(b)^j e_j \end{aligned}$$

That is, for $j = 0, 1, 2, \cdots$, the vectors e_j are eigenvectors for $W_{f,\varphi}$ with eigenvalues $\overline{f(b)}\varphi'(b)^j$. Since $W_{f,\varphi} = T_f C_\varphi$ is compact and the vectors e_j form an orthonormal basis for H^2, the spectrum of $W_{f,\varphi}$ is

$$\sigma(W_{f,\varphi}) = \{0\} \cup \{f(b), f(b)\varphi'(b), \cdots, f(b)(\varphi'(b))^j, \cdots\}.$$

To understand these eigenvalues better, we will calculate $\overline{f(b)}$. If b is the fixed point of φ that is in the open unit disk, then Lemma 4 and the assumptions on a_1 imply

$$b = \frac{1 + |a_0|^2 - a_1 - \sqrt{(1 + |a_0|^2 - a_1)^2 - 4|a_0|^2}}{2\overline{a_0}}$$

Thus,

$$\begin{aligned} \overline{f(b)} &= \overline{\frac{c}{1-\overline{a_0}b}} = \overline{\frac{c}{1 - \left(1 + |a_0|^2 - a_1 - \sqrt{(1 + |a_0|^2 - a_1)^2 - 4|a_0|^2}\right)/2}} \\ &= \frac{2c}{1 + a_1 - |a_0|^2 + \sqrt{(1 + |a_0|^2 - a_1)^2 - 4|a_0|^2}} \end{aligned}$$

In particular, $\overline{f(b)}$ is a real number which is consistent with the fact that $W_{f,\varphi}$ is Hermitian.

The following result formalizes the work of this section.

Theorem 9. *Let a_0 be a point of the open unit disk, $a_0 \neq 0$. If $-1 + |a_0|^2 < a_1 < (1 - |a_0|)^2$, then the weighted composition operator $T_f C_\varphi$ is compact on H^2. Moreover, if b is the fixed point of φ in the open unit disk, then the set $\{e_j : j = 0, 1, 2, \cdots\}$ is an orthonormal basis for H^2 consisting of eigenvectors for $W_{f,\varphi} = T_f C_\varphi$, where*

$$e_j(z) = \frac{\sqrt{1 - |b|^2}}{1 - \bar{b}z} \left(\frac{z - b}{\bar{b}z - 1} \right)^j .$$

The spectrum of $W_{f,\varphi}$ is

$$\sigma(W_{f,\varphi}) = \{0\} \cup \{f(b), f(b)\varphi'(b), \cdots, f(b)(\varphi'(b))^j, \cdots\}.$$

5. Absolutely continuous Hermitian weighted composition operators

To study the remaining cases of Hermitian weighted composition operators, when $a_1 = (1 - |a_0|)^2$ for $0 < |a_0| < 1$, we first show that the operators $W_{f,\varphi}$ belong to a continuous semigroup of Hermitian operators. Recall that an indexed collection $\{A_t : t \geq 0\}$ of bounded operators is called a continuous semigroup of operators if $A_{s+t} = A_s A_t$ for all non-negative real numbers s and t, $A_0 = I$, and the map $t \mapsto A_t$ is strongly continuous. Similarly, for $0 < \theta \leq \frac{\pi}{2}$, an indexed collection $\{A_t : |\arg t| < \theta\}$ of bounded operators is called a holomorphic semigroup of operators if $A_{s+t} = A_s A_t$ for $|\arg s| < \theta$ and $|\arg t| < \theta$ and the map $t \mapsto A_t$ is holomorphic.

Notice that if $W_{f,\varphi}$ and $W_{g,\psi}$ are weighted composition operators, then

$$\begin{aligned} W_{f,\varphi} W_{g,\psi}(h) &= T_f C_\varphi T_g C_\psi(h) = T_f C_\varphi T_g (h \circ \psi) = T_f C_\varphi (g \cdot (h \circ \psi)) \\ &= T_f ((g \circ \varphi) \cdot (h \circ (\psi \circ \varphi))) = f \cdot (g \circ \varphi) \cdot (h \circ (\psi \circ \varphi)) \\ &= \left(T_{f \cdot (g \circ \varphi)} C_{\psi \circ \varphi} \right)(h) = W_{f \cdot (g \circ \varphi), \psi \circ \varphi}(h) \end{aligned}$$

so that

$$W_{f,\varphi} W_{g,\psi} = W_{f \cdot (g \circ \varphi), \psi \circ \varphi}$$

Now suppose that $\{T_{f_t} C_{\varphi_t}\}$ is a semigroup of weighted composition operators. Then the semigroup law is

$$(T_{f_s} C_{\varphi_s})(T_{f_t} C_{\varphi_t}) = T_{f_{s+t}} C_{\varphi_{s+t}}$$

or from the above calculation,

$$T_{f_s \cdot (f_t \circ \varphi_s)} C_{\varphi_t \circ \varphi_s} = T_{f_{s+t}} C_{\varphi_{s+t}}$$

Evaluating this equality at the function 1 in H^2, we get

$$T_{f_s \cdot (f_t \circ \varphi_s)} C_{\varphi_t \circ \varphi_s}(1) = f_s \cdot (f_t \circ \varphi_s) \cdot (1 \circ (\varphi_t \circ \varphi_s)) = f_s \cdot (f_t \circ \varphi_s)$$

and

$$T_{f_{s+t}} C_{\varphi_{s+t}}(1) = f_{s+t} \cdot (1 \circ \varphi_{s+t}) = f_{s+t}$$

which means

$$f_s \cdot (f_t \circ \varphi_s) = f_{s+t} \tag{9}$$

for all s and t. Similarly, evaluating at the identity function $\chi(z) = z$ in H^2, we get

$$T_{f_s\cdot(f_t\circ\varphi_s)}C_{\varphi_t\circ\varphi_s}(\chi) = f_s\cdot(f_t\circ\varphi_s)\cdot(\chi\circ(\varphi_t\circ\varphi_s)) = f_s\cdot(f_t\circ\varphi_s)\cdot(\varphi_t\circ\varphi_s)$$

and

$$T_{f_{s+t}}C_{\varphi_{s+t}}(\chi) = f_{s+t}\cdot(\chi\circ\varphi_{s+t}) = f_{s+t}\varphi_{s+t}$$

which means

$$f_s\cdot(f_t\circ\varphi_s)\cdot(\varphi_t\circ\varphi_s) = f_{s+t}\varphi_{s+t}$$

from which it follows that

$$\varphi_t\circ\varphi_s = \varphi_{s+t} \tag{10}$$

In other words, the composition operator factors in a semigroup of weighted composition operators form a semigroup of composition operators and the Toeplitz operator factors form a 'cocycle' of Toeplitz operators.

For $\mathrm{Re}\, t > 0$, let $A_t = T_{f_t}C_{\varphi_t}$ where

$$f_t(z) = \frac{1}{1+t-tz}$$

and

$$\varphi_t(z) = \frac{t+(1-t)z}{1+t-tz}$$

Note that the relationship between a_0 and t can be expressed as $a_0 = \varphi(0) = t/(1+t)$ for $|a_0 - \frac{1}{2}| < \frac{1}{2}$. However, $a_1 = \varphi'(0) = (1+t)^{-2}$ is not real unless t is real, so A_t is not Hermitian for t not real.

Theorem 10. *The A_t, for $\mathrm{Re}\, t > 0$, form a holomorphic semigroup of weighted composition operators.*

Proof. To show $A_tA_s = A_{t+s}$, it suffices to show that the cocycle relationship (Equation (9)) and the semigroup relationship (Equation (10)) hold. For the f_t and φ_t given above, the required equalities are

$$f_s(z)\cdot f_t(\varphi_s(z)) = \frac{1}{1+s-sz}\frac{1}{1+t-t\frac{s+(1-s)z}{1+s-sz}} = \frac{1}{1+(s+t)-(s+t)z} = f_{s+t}(z)$$

and

$$\varphi_s(\varphi_t(z)) = \frac{s+(1-s)\frac{t+(1-t)z}{1+t-tz}}{1+s-s\frac{t+(1-t)z}{1+t-tz}} = \frac{(s+t)+(1-(s+t))z}{1+(s+t)-(s+t)z} = \varphi_{s+t}(z)$$

Thus, the set $\{A_t : \mathrm{Re}\, t > 0\}$ is a semigroup of weighted composition operators.

Since operator valued functions are analytic in the norm topology if and only if they are analytic in the weak-operator topology (Theorem 3.10.1 of [11, p. 93]), it is sufficient to check that the map $t \mapsto \langle A_th, K_z\rangle$ is holomorphic for each t in the right half plane. This is easy to see because h is holomorphic and

$$\langle A_th, K_z\rangle = f_t(z)h(\varphi_t(z)) = \frac{1}{1+t-tz}h\left(\frac{t+(1-t)z}{1+t-tz}\right)$$

which is clearly holomorphic in t for fixed z. ☐

To consider the Hermitian case, by normalizing using Corollary 2, we may assume $0 < a_0 < 1$. That is, Corollary 2 says that each instance of this case, $a_1 = (1 - |a_0|)^2$ for $0 < |a_0| < 1$, the operator $W_{f,\varphi}$ is unitarily equivalent to exactly one $W_{g,\psi}$ for which $\psi(0) > 0$, namely the one for which $\psi(0) = |a_0| > 0$ and $\psi'(0) = a_1 = (1 - |a_0|)^2 = (1 - \psi(0))^2$. Writing $t = a_0/(1 - a_0)$, each such $W_{f,\varphi} = T_f C_\varphi$ Hermitian weighted composition operator in the normalized third case, that is, $0 < a_0 < 1$ and $a_1 = (1 - a_0)^2$, is a multiple of $A_t = T_{f_t} C_{\varphi_t}$ where f_t and φ_t are as above. For $0 < a_0 < 1$, it is easy to see that $0 < t < \infty$. We show that the A_t for $0 \leq t < \infty$ form a semigroup of Hemitian weighted composition operators.

In his thesis [15, p. 28], H. Sadraoui showed that $\|A_t\| = 1$ for each t with $0 \leq t < \infty$ and this also follows from Theorem 13 or Corollary 14, whose proofs do not depend on the result.

Corollary 11. *The A_t, for $0 \leq t < \infty$, are a continuous semigroup of Hermitian weighted composition operators.*

Proof. We see that $A_0 = 1C_z = I$. For $0 \leq t < \infty$, the symbols f_t and φ_t satisfy the conditions of Theorem 5, so each of the operators A_t is Hermitian. Since Theorem 10 shows that $\{A_t : \mathrm{Re}\, t > 0\}$ is a semigroup, the semi-group property is true for $t \geq 0$.

Furthermore, for any α in $\mathbb{D}$,

$$\begin{aligned}
\lim_{t \to 0+} A_t K_\alpha &= \lim_{t \to 0+} C_{\varphi_t}^* T_{f_t}^* K_\alpha \\
&= \lim_{t \to 0+} \overline{f_t(\alpha)} K_{\varphi_t}(\alpha) \\
&= \lim_{t \to 0+} \frac{1}{-\overline{\alpha} t + (1 + t) - [(1 - t)\overline{\alpha} + t] z} \\
&= K_\alpha
\end{aligned}$$

Since the kernel functions have dense span in H^2 and $\|A_t\| = 1$ for each $t \geq 0$, it follows that for each f in H^2, we also have $\lim_{t \to 0+} A_t f = f$. Thus, A_t is strongly continuous and the proof is complete. □

The following result provides a foundation for one version of the Spectral Theorem for the Hermitian weighted composition operators A_t.

Theorem 12. *For $0 < t < \infty$, each A_t is a (star) cyclic Hermitian operator. In particular, the vector 1 in H^2 is a (star-)cyclic vector for A_t.*

Proof. We note that because A_t is Hermitian, a vector is a star-cyclic vector exactly when it is a cyclic vector.

For $0 < t < \infty$ and α in the unit disk,

$$A_t K_\alpha = C_{\varphi_t}^* T_{f_t}^* K_\alpha = \overline{f_t(\alpha)} K_{\varphi_t(\alpha)} = f_t(\overline{\alpha}) K_{\varphi_t(\alpha)}$$

Since the vector 1 in H^2 is K_0, we have $A_t(1) = f_t(0) K_{\varphi_t(0)}$. Now,

$$A_t \left(f_t(0) K_{\varphi_t(0)} \right) = A_t(A_t(1)) = A_{2t}(1) = f_{2t}(0) K_{\varphi_{2t}(0)}$$

and in general, clearly, $A_t^n(1) = f_{nt}(0) K_{\varphi_{nt}(0)}$.

To check cyclicity, we need to investigate the span of these vectors. Since the factor $f_{nt}(0)$ is just a non-zero number, 1 is a cyclic vector for A_t if and only if span$\{K_{\varphi_{nt}(0)}\}$ is dense in H^2. This span is dense if and only if the only vector orthogonal to all the vectors $K_{\varphi_{nt}(0)}$ is 0. Since $\langle h, K_{\varphi_{nt}(0)}\rangle = h(\varphi_{nt}(0))$, this means that the span is dense if and only if the only function h in H^2 such that $h(\varphi_{nt}(0)) = 0$ for $n = 1, 2, 3, \cdots$ is the zero function.

That is, the span fails to be dense if the sequence $\{\varphi_{nt}(0)\}$ is a Blaschke sequence and is dense if the sequence $\{\varphi_{nt}(0)\}$ is not a Blaschke sequence. Now, consider the sum

$$\sum_{n=1}^{\infty}(1 - |\varphi_{nt}(0)|) = \sum_{n=1}^{\infty}\left(1 - \frac{nt}{1+nt}\right) = \sum_{n=1}^{\infty}\frac{1}{1+nt} = \infty$$

Since this sum is infinite, the sequence is *not* a Blaschke sequence and therefore the vectors $\{K_{\varphi_{nt}(0)}\}$ have dense span and 1 is a cyclic vector for A_t. □

This means that each of these operators is unitarily equivalent to an ordinary multiplication operator (see, for example, [1, p. 269]). The following result gives this explicitly.

In the theorem below, the equation $(M_h f)(x) = h(x)f(x)$ for f in $L^2([0,1], dx)$ defines M_h on L^2.

Theorem 13. *For $0 < t < \infty$, the Hermitian weighted composition operator A_t is unitarily equivalent to M_{x^t} on $L^2([0,1], dx)$. In fact, the operator $U : H^2 \to L^2$ given by*

$$U(f_s(0)K_{\varphi_s(0)}) = x^s$$

is unitary and $UA_t = M_{x^t}U$.

Proof. Let t be a positive real number. In the proof of Theorem 12, we saw that $1 = f_0(0)K_{\varphi_0(0)}$ is a cyclic vector for A_t, that

$$A_t^n(f_0(0)K_{\varphi_0(0)}) = f_{tn}(0)K_{\varphi_{tn}(0)}$$

and that the latter set, for $n = 0, 1, 2, \cdots$, has dense span in H^2. Since the operator M_{x^t} is a bounded Hermitian operator on $L^2([0,1], dx)$, we see easily that $M_{x^t}^n 1 = x^{tn}$, and the Stone-Weierstrass Approximation Theorem shows that $\{x^{tn}\}_{n=0,1,2,\cdots}$, has dense span in $L^2([0,1], dx)$. Thus, if we define U as above, we see that for each non-negative integer n,

$$UA_t(f_n(0)K_{\varphi_n(0)}) = M_{x^t}U(f_n(0)K_{\varphi_n(0)})$$

We will show that U is isometric on the span of $\{f_s(0)K_{\varphi_s(0)}\}_{0\leq s<\infty}$. For $0 \leq r, s < \infty$, on H^2,

$$\begin{aligned}
\langle f_r(0)K_{\varphi_r(0)}, f_s(0)K_{\varphi_s(0)}\rangle &= \langle \frac{1}{1+r}\frac{1}{1-\frac{r}{1+r}z}, \frac{1}{1+s}\frac{1}{1-\frac{s}{1+s}z}\rangle \\
&= \frac{1}{1+s}\langle \frac{1}{1+r-rz}, \frac{1}{1-\frac{s}{1+s}z}\rangle \\
&= \frac{1}{1+s}\frac{1}{1+r-r\frac{s}{1+s}} = \frac{1}{(1+s)(1+r)-rs} \\
&= \frac{1}{1+r+s}
\end{aligned}$$

Similarly, for $0 \leq r, s < \infty$, on $L^2([0,1], dx)$,

$$\langle x^r, x^s\rangle_{L^2} = \int_0^1 x^r x^s\, dx = \int_0^1 x^{r+s}\, dx = \frac{1}{1+r+s}x^{1+r+s}\Big|_0^1 = \frac{1}{1+r+s}$$

Now, the vectors $\{f_s(0)K_{\varphi_s(0)}\}_{0<s<1}$ are linearly independent in H^2 and have dense span. Similarly, the vectors $\{x^s\}_{0<s<1}$ are linearly independent and have dense span in $L^2([0,1], dx)$. It follows that U is well defined as a linear transformation from span$\{f_s(0)K_{\varphi_s(0)}\}_{0<s<1}$ in H^2 to span$\{x^s\}_{0<s<1}$ in $L^2([0,1], dx)$. The inner product calculations above show that U is isometric on these spans and therefore has a unique extension to an isometric operator from the closure of span$\{f_s(0)K_{\varphi_s(0)}\}_{0<s<1}$ onto the closure of span$\{x^s\}_{0<s<1}$ in $L^2([0,1], dx)$, that is, U is an isometric operator of H^2 onto $L^2([0,1], dx)$, that is, it is a unitary operator between these spaces. □

Corollary 14. *For $0 < t < \infty$, each A_t is unitarily equivalent to multiplication by y in $L^2([0,1], \mu)$ for the Borel measure*

$$\mu(dy) = \frac{1}{t}y^{\frac{1}{t}-1}dy$$

Proof. Theorem 13 shows that A_t is unitarily equivalent to multiplication by x^t on $L^2([0,1], dx)$, so it is enough prove the unitary equivalence of multiplication by x^t on $L^2([0,1], dx)$ with multiplication by y on $L^2([0,1], \mu)$.

The unitary operator is just the change of variables map defined by

$$V : L^2([0,1], \mu) \to L^2([0,1], dx) \;\; \text{defined by} \;\; (Vf)(x) = f(x^t)$$

for f in $L^2([0,1], \mu)$ where μ is the measure in the statement of the Corollary. □

In the remainder of this section, we study the properties of the semigroup so that we can identify the spectral measures of the A_t in a more concrete way than the unitary operators of the above results.

Recall that the infinitesimal generator Δ of $\{A_t : t \geq 0\}$ is the operator defined by

$$\Delta f = \lim_{t\to 0^+} \frac{1}{t}(A_t - I)f \quad \text{for } f \text{ in } H^2$$

Theorem 15. *The infinitesimal generator of the semigroup* $\{A_t\}$ *is*

$$\Delta(f)(z) = (z-1)((z-1)f(z))'.$$

Proof. Let f be a function in H^2 and suppose z is a point of the disk. Then we have

$$\begin{aligned}
(\Delta f)(z) &= \lim_{t\to 0^+} \frac{1}{t}\left((A_t - I)f\right)(z) = \lim_{t\to 0^+} \frac{f_t(z)f(\varphi_t(z)) - f(z)}{t} \\
&= \lim_{t\to 0^+} \frac{f_t(z)f(\varphi_t(z)) - f_t(z)f(z) + f_t(z)f(z) - f(z)}{t} \\
&= \lim_{t\to 0^+} f_t(z)\frac{f(\varphi_t(z)) - f(z)}{\varphi_t(z)-z}\frac{\varphi_t(z)-z}{t} + \frac{f_t(z)-1}{t}f(z) \\
&= 1\cdot f'(z)\cdot(1-z)^2 - (1-z)f(z) = (z-1)\left((z-1)f'(z) + f(z)\right) \\
&= (z-1)\left((z-1)f(z)\right)'
\end{aligned}$$

□

Lemma 16. *If* $\Delta(f)(z) = \lambda f(z)$*, then*

$$f_\lambda(z) = \frac{\kappa}{1-z}e^{\frac{\lambda}{1-z}}$$

for some constant κ*.*

Proof. Since $\Delta(f)(z) = (z-1)((z-1)f(z))'$, we get

$$(z-1)((z-1)f(z))' = \lambda f(z).$$

Then

$$(z-1)\{f(z) + (z-1)f'(z)\} = \lambda f(z).$$

Hence

$$(z-1)^2 f'(z) = -(z-1-\lambda)f(z).$$

Thus

$$\frac{f'(z)}{f(z)} = -\frac{1}{z-1} + \frac{\lambda}{(z-1)^2}.$$

Integrating both sides, we get

$$\ln f(z) = -\ln(z-1) - \lambda\frac{1}{z-1} + c$$

for some constant c. Hence, for $\kappa = e^c$, we get

$$f(z) = \frac{1}{1-z}e^{\frac{\lambda}{1-z}}e^c = \frac{\kappa}{1-z}e^{\frac{\lambda}{1-z}}.$$

□

Lemma 17. *For* $1 \le p \le \infty$*, there is no* λ *for which* $f_\lambda(z) = \dfrac{\kappa}{1-z}e^{\frac{\lambda}{1-z}}$ *is in* H^p*.*

Proof. Notice that

$$e^{\frac{\lambda}{1-z}} = e^{\frac{\lambda}{2}} e^{\frac{\lambda}{2}\left(\frac{1+z}{1-z}\right)}$$

so the radial limits of $|e^{\frac{\lambda}{1-z}}|$ are equal to $e^{\frac{\lambda}{2}}$ for every radius except the radius along the positive real axis.

Since f_λ in H^p implies its radial limit function is in $L^p(\partial\mathbb{D})$ and

$$\lim_{r\to 1^-} |f_\lambda(re^{i\theta})| = \frac{\kappa e^{\frac{\lambda}{2}}}{|1-e^{i\theta}|}$$

is not in L^p for $p \geq 1$, we see that f_λ is not in H^p for $p \geq 1$. □

Corollary 18. *For $t > 0$, the operator A_t has no eigenvalues.*

Proof. Because $\{A_t\}_{t\geq 0}$ is a strongly continuous semigroup,

$$\sigma_p(A_t) \subset e^{t\sigma_p(\Delta)} \cup \{0\}$$

(See, for example, [14, p. 46].) Because $A_t = T_{f_t} C_{\varphi_t}$ and both the Toeplitz operator and the composition operator are one-to-one, 0 is not in the point spectrum of A_t. Also we know that λ is in $\sigma_p(\Delta)$ if and only if

$$f_\lambda(z) = \frac{\kappa}{1-z} e^{\frac{\lambda}{1-z}} \in H^2$$

By Lemma 17, $\sigma_p(\Delta) = \emptyset$, which completes the proof. □

Thus, we have the situation that the vectors we would expect to be the eigenvectors of the operators in the semigroup are not in $H^2$and the operators, in fact, have no eigenvectors. The following result is a motivated by the calculations above and is a substitute for the failure of the operators to have eigenvectors.

Theorem 19. *Let $g_s(z) = (s-z)^{-1} e^{\frac{\lambda}{1-z}}$ for $\lambda < 0$ and $s > 1$. Then*

$$\lim_{s\to 1} \| A_t \frac{g_s}{\|g_s\|} - e^{t\lambda} \frac{g_s}{\|g_s\|} \| = 0$$

and for $\lambda < 0$, this means $e^{t\lambda}$ is in the approximate point spectrum, $\sigma_{ap}(A_t)$.

Proof.

$$\begin{aligned}
A_t g_s(z) &= T_{f_t} C_{\varphi_t} g_s(z) \\
&= T_{f_t} g_s(f_t(z)) \\
&= T_{f_t} \frac{1}{s-\varphi_t(z)} e^{\frac{\lambda}{1-\varphi_t(z)}} \\
&= f_t(z) \frac{1}{s-\varphi_t(z)} e^{\frac{\lambda[1+t-tz]}{1-z}} \\
&= \frac{1}{s(1+t-tz)-(t+(1-t)z)} e^{t\lambda} e^{\frac{\lambda}{1-z}}
\end{aligned}$$

Fix $s > 1$. Then

$$
\begin{aligned}
&\|A_t g_s(z) - e^{t\lambda} g_s(z)\| \\
&= \left\| \{\frac{1}{s(1+t-tz)-(t+(1-t)z)} - \frac{1}{s-z}\} e^{t\lambda} e^{\frac{\lambda}{1-z}} \right\| \\
&\leq (s-1) \left\| \frac{t}{z(-ts+t-1)-(-ts+t-s)} e^{t\lambda} e^{\frac{\lambda}{1-z}} \right\| \\
&\leq (s-1) \left\| \frac{t}{-z(ts-t+1)+(ts-t+s)} \right\| \\
&= \frac{s-1}{|ts-t+s|} \left\| \frac{1}{1-\frac{ts-t+1}{ts-t+s}z} \right\| \\
&= \frac{s-1}{|ts-t+s|} \left\| K_{\frac{ts-t+1}{ts-t+s}} \right\| \\
&= \frac{s-1}{|ts-t+s|} \left(1 - \left| \frac{ts-t+1}{ts-t+s} \right|^2 \right)^{-\frac{1}{2}} \\
&= \frac{(s-1)t}{\sqrt{(s-1)[s+1+2t(s-1)]}}
\end{aligned}
$$

On the other hand, since $e^{-\frac{1+z}{1-z}}$ is a singular inner function,

$$
\begin{aligned}
\|g_s(z)\|^2 &= \langle \frac{1}{s-z} e^{\frac{\lambda}{1-z}}, \frac{1}{s-z} e^{\frac{\lambda}{1-z}} \rangle \\
&= \langle \frac{1}{s-z} e^{\frac{\lambda}{2}} e^{\frac{\lambda}{2}(\frac{1+z}{1-z})}, \frac{1}{s-z} e^{\frac{\lambda}{2}} e^{\frac{\lambda}{2}(\frac{1+z}{1-z})} \rangle \\
&= e^{\lambda} \langle \frac{1}{s-z} \left(e^{-\frac{1+z}{1-z}} \right)^{-\frac{\lambda}{2}}, \frac{1}{s-z} \left(e^{-\frac{1+z}{1-z}} \right)^{-\frac{\lambda}{2}} \rangle \\
&= e^{\lambda} \langle \frac{1}{s-z}, \frac{1}{s-z} \rangle \\
&= \frac{e^{\lambda}}{s^2} \langle K_{\frac{1}{s}}, K_{\frac{1}{s}} \rangle \\
&= \frac{e^{\lambda}}{s^2} K_{\frac{1}{s}}(\frac{1}{s}) \\
&= e^{\lambda} \frac{1}{s^2-1}
\end{aligned}
$$

Hence

$$\|g_s\| = \frac{e^{\frac{\lambda}{2}}}{\sqrt{s^2-1}}.$$

Therefore,

$$
\begin{aligned}
&\lim_{s\to 1} \|A_t \frac{g_s}{\|g_s\|} - e^{t\lambda} \frac{g_s}{\|g_s\|}\| \\
\leq\ &\lim_{s\to 1} (s-1) t e^{\frac{\lambda}{2}} \frac{\sqrt{s^2-1}}{\sqrt{(s-1)[s+1+2t(s-1)]}} = 0.
\end{aligned}
$$

Thus for $\lambda < 0$, the number $e^{t\lambda}$ is in $\sigma_{ap}(A_t)$. □

Corollary 20. *For each $t \geq 0$, the spectrum of A_t is given by $\sigma(A_t) = [0, 1]$.*

Proof. For $0 < t < \infty$, from the semigroup property, we have $A_t = A_{t/2}^2$. It follows that each A_t is a positive operator and $\sigma(A_t) \subset [0, \infty)$.

By Theorem 19, we get

$$\{e^{t\lambda} : \lambda < 0\} \subset \sigma_{ap}(A_t) \subset \sigma(A_t)$$

Since $\{e^{t\lambda} : \lambda < 0\} = (0, 1)$ and $\sigma(A_t)$ is compact,

$$[0, 1] \subset \sigma(A_t).$$

Finally, the fact that $\|A_t\| = 1$ shows that $\sigma(A_t) = [0, 1]$. □

Note: a different computation of the spectra can be given using fact that these operators are part of a holomorphic semigroup and applying the Gelfand theory as in [4, p. 102] or [7, p. 302]. Furthermore, the unitary equivalences of Theorem 13 and Corollary 14

Recall the following property of isometries on Hilbert spaces.

Lemma 21. *If S is an isometry, that is, S is a bounded operator such that $S^*S = I$, then SS^* is the orthogonal projection onto range of S.*

The idea underlying the computation of the spectral measure for our operators is that the eigenvectors of our operators should be

$$\frac{1}{1-z}e^{\frac{\lambda}{1-z}} = \frac{e^{\frac{\lambda}{2}}}{1-z}e^{\frac{\lambda}{2}\frac{1+z}{1-z}}$$

If these were eigenvectors for A_1, say, then the spectral measure associated with $[0, r]$ for $0 \leq r \leq 1$ would be the projection onto the subspace spanned by eigenvectors whose eigenvalues are in $[0, r]$. This will be the case for A_1 if and only if λ is a number so that $0 < e^{\lambda} \leq r$. So suppose r is given and $\lambda_0 = \ln r$ so that $e^{\lambda_0} = r$. Looking at the "eigenvectors", it looks like the subspace containing the eigenvectors for the eigenvalues with $0 < e^{\lambda} < r$ ought, if they were actually in H^2, to span the subspace $e^{\frac{\lambda_0}{2}\frac{1+z}{1-z}}H^2$.

We want to prove that this is the correct set of projections. For $\lambda \leq 0$, the Toeplitz operator

$$T_{e^{\frac{\lambda}{2}\frac{1+z}{1-z}}}$$

is an isometry because $e^{\frac{\lambda}{2}\frac{1+z}{1-z}}$ is an inner function. For $0 < r \leq 1$, by Lemma 21,

$$P_r = T_{e^{\frac{\ln r}{2}\frac{1+z}{1-z}}}T^*_{e^{\frac{\ln r}{2}\frac{1+z}{1-z}}}$$

is the orthogonal projection onto the range of the inner function Toeplitz operator which is just $e^{\frac{\ln r}{2}\frac{1+z}{1-z}}H^2$. Thus, we want to prove that the family of projections in the spectral measure of our operators is

$$\{P_r : 0 < r \leq 1\}$$

We begin by showing that these projections have the properties appropriate for creating a spectral measure.

Proposition 22. *For $0 < r < s \le 1$, $P_r P_s = P_s P_r = P_r$. In particular, the projections P_r and P_s commute. Furthermore, $P_1 = I$ and $\cap_{0<r\le 1} \text{ran}\, P_r = (0)$.*

Proof. Since $\ln 1 = 0$, the Toeplitz operator $T_{e^{\frac{\ln 1}{2}\frac{1+z}{1-z}}} = T_{e^0} = T_1 = I$, so $P_1 = I$.

Suppose f is in $\cap_{0<r\le 1}\text{ran}\, P_r$ and $f = ug$ is its inner – outer factorization. For a point α in the open disk, we have $|f(\alpha)| = |u(\alpha)||g(\alpha)|$. Since f is in ran $P_{1/e}$, we know that the inner function $e^{-\frac{1}{2}\frac{1+z}{1-z}}$ divides u and $|u(\alpha)| \le |e^{-\frac{1}{2}\frac{1+\alpha}{1-\alpha}}| < 1$. In the same way, we see, since f is in the range of $P_{e^{-n}}$, that

$$|u(\alpha)| \le |e^{-\frac{n}{2}\frac{1+\alpha}{1-\alpha}}|$$

for every positive integer n. It follows that $u(\alpha) = 0$ but since this is true for each α in the disk, $u \equiv 0$ and therefore $f = ug \equiv 0$ also. This shows $\cap_{0<r\le 1}\text{ran}\, P_r = (0)$.

We claim that for $r < s$, we have $\text{ran}\, P_r \subset \text{ran}\, P_s$. If f is in $\text{ran}\, P_r$, then there is a function g in H^2 such that $f = e^{\frac{\ln r}{2}\frac{1+z}{1-z}} g$. Now

$$f = e^{\frac{\ln r}{2}\frac{1+z}{1-z}} g = e^{\frac{\ln s}{2}\frac{1+z}{1-z}} \left(e^{(\frac{\ln r}{2} - \frac{\ln s}{2})\frac{1+z}{1-z}} g \right) \tag{11}$$

Since $\ln r < \ln s$, we have $\ln r - \ln s < 0$ which means $e^{(\frac{\ln r}{2} - \frac{\ln s}{2})\frac{1+z}{1-z}}$ is an inner function and the latter factor above,

$$e^{(\frac{\ln r}{2} - \frac{\ln s}{2})\frac{1+z}{1-z}} g$$

is in H^2. Therefore, Equation (11) says that f is a product of $e^{\frac{\ln s}{2}\frac{1+z}{1-z}}$ and a function in H^2, which means it is in the range of P_s. Because f was an arbitrary function in the range of P_r, we have $\text{ran}\, P_r \subset \text{ran}\, P_s$.

Since P_r and P_s are orthogonal projections and $\text{ran}\, P_r \subset \text{ran}\, P_s$, we have both $P_r P_s = P_r$ and $P_s P_r = P_r$, as we wished to prove. □

The following lemma will facilitate our calculations involving the projections of interest and the semi-group operators.

Lemma 23. *If u is an inner function, $P = T_u T_u^*$ is the projection onto uH^2, and α is a point of the unit disk, then*

$$PK_\alpha = \overline{u(\alpha)} u K_\alpha$$

Proof.

$$\begin{aligned} (PK_\alpha)(z) &= \left(T_u T_u^* K_\alpha\right)(z) = \left(T_u \overline{u(\alpha)} K_\alpha\right)(z) \\ &= \overline{u(\alpha)} \left(u K_\alpha\right)(z) = \overline{u(\alpha)} u(z) K_\alpha(z) \end{aligned}$$

□

In particular, this calculation allows us to show that the projections commute the semi-group operators.

Proposition 24. *For $0 < r \le 1$, each of the projections P_r commutes with A_t.*

Proof. Let u be the inner function $u(z) = e^{\frac{\ln r}{2}\frac{1+z}{1-z}}$ so that P_r is the projection onto uH^2. We keep in mind that A_t is Hermitian, so that $A_t = T_{f_t} C_{\varphi_t} = C_{\varphi_t}^* T_{f_t}^*$.

For any α in $\mathbb{D}$, using Lemma 23,

$$\begin{aligned} \left(P_r A_t K_\alpha\right)(z) &= \left(P_r C_{\varphi_t}^* T_{f_t}^* K_\alpha\right)(z) = \overline{f_t(\alpha)}\left(P_r K_{\varphi_t(\alpha)}\right)(z) \\ &= \overline{f_t(\alpha) u(\varphi_t(\alpha))} u(z) K_{\varphi_t(\alpha)}(z) \end{aligned}$$

On the other hand, using Lemma 23 and Proposition 3,

$$\begin{aligned} \left(A_t P_r K_\alpha\right)(z) &= \overline{u(\alpha)}\left(A_t u K_\alpha\right)(z) = \overline{u(\alpha)} u(\varphi_t(z))\left(A_t K_\alpha\right)(z) \\ &= \overline{u(\alpha)} u(\varphi_t(z))\left(C_{\varphi_t}^* T_{f_t}^* K_\alpha\right)(z) = \overline{u(\alpha)} u(\varphi_t(z)) \overline{f_t(\alpha)} K_{\varphi_t(\alpha)}(z) \end{aligned}$$

Comparing the two expressions, we see that we must consider $\overline{u(\alpha)} u(\varphi_t(z))$ and $\overline{u(\varphi_t(\alpha))} u(z)$. Note that because the Taylor coefficients of the various functions involved are real, this is the same as comparing $u(\overline{\alpha}) u(\varphi_t(z))$ and $u(\varphi_t(\overline{\alpha})) u(z)$. A tedious calculation shows that

$$u(\overline{\alpha}) u(\varphi_t(z)) = u(\varphi_t(\overline{\alpha})) u(z) = e^{t \ln r} u(z) u(\overline{\alpha})$$

and we conclude that $P_r A_t K_\alpha = A_t P_r K_\alpha$ for every α in the disk. Since the span of the K_α is dense in H^2 and the operators are bounded, we see that $P_r A_t = A_t P_r$ as desired. □

Theorem 25. *Let t be a positive real number. Letting $P_0 = 0$, the projections $\{P_r\}_{0 \le r \le 1}$ form a resolution of the identity for the operator A_t. Related to A_t, the projection P_r corresponds to the interval $[0, r^t]$ as a subset of the spectrum of A_t. This means we have*

$$A_t = \int_0^1 r^t \, dP_r$$

Proof. For $0 < r \le 1$, let u_r be the inner function $u_r(z) = e^{\frac{\ln r}{2} \frac{1+z}{1-z}}$ so that P_r is the projection onto $u_r H^2$. Lemma 23 says that, for α a point of the unit disk,

$$(P_r K_\alpha)(z) = \overline{u_r(\alpha)} u_r(z) K_\alpha(z)$$

If $0 = r_0 < r_1 < r_2 < \cdots < r_{n-1} < r_n = 1$, then, for $1 < j \le n$,

$$\begin{aligned} \left((P_{r_j} - P_{r_{j-1}}) K_\alpha\right)(z) &= \left(P_{r_j} K_\alpha\right)(z) - \left(P_{r_{j-1}} K_\alpha\right)(z) \\ &= \overline{u_{r_j}(\alpha)} u_{r_j}(z) K_\alpha(z) - \overline{u_{r_{j-1}}(\alpha)} u_{r_{j-1}}(z) K_\alpha(z) \\ &= \left(e^{\frac{\ln r_j}{2}\left(\frac{1+\overline{\alpha}}{1-\overline{\alpha}} + \frac{1+z}{1-z}\right)} - e^{\frac{\ln r_{j-1}}{2}\left(\frac{1+\overline{\alpha}}{1-\overline{\alpha}} + \frac{1+z}{1-z}\right)}\right) K_\alpha(z) \end{aligned}$$

(When $j = 1$, easy adjustments in the formulas must be made because $P_0 = 0$.)

Combining these into a single sum, we have

$$\begin{aligned} &\sum e^{t \ln r_j}\left((P_{r_j} - P_{r_{j-1}}) K_\alpha\right)(z) \\ &= \left(\sum e^{t \ln r_j}\left(e^{\frac{\ln r_j}{2}\left(\frac{1+\overline{\alpha}}{1-\overline{\alpha}} + \frac{1+z}{1-z}\right)} - e^{\frac{\ln r_{j-1}}{2}\left(\frac{1+\overline{\alpha}}{1-\overline{\alpha}} + \frac{1+z}{1-z}\right)}\right)\right) K_\alpha(z) \\ &= \left(\sum e^{t \ln r_j} \frac{e^{\frac{\ln r_j}{2}\left(\frac{1+\overline{\alpha}}{1-\overline{\alpha}} + \frac{1+z}{1-z}\right)} - e^{\frac{\ln r_{j-1}}{2}\left(\frac{1+\overline{\alpha}}{1-\overline{\alpha}} + \frac{1+z}{1-z}\right)}}{r_j - r_{j-1}} (r_j - r_{j-1})\right) K_\alpha(z) \end{aligned}$$

Hence $\sum e^{t\ln r_j}\left((P_{r_j}-P_{r_{j-1}})K_\alpha\right)(z)$ converges to the following integral.

$$\int_0^1 r^t \frac{d}{dr}[e^{\frac{\ln r}{2}\left(\frac{1+\overline{\alpha}}{1-\overline{\alpha}}+\frac{1+z}{1-z}\right)}]\, dr K_\alpha(z)$$

$$\begin{aligned}
&= \int_0^1 r^t \frac{d}{dr}[r^{\frac{1}{2}\left(\frac{1+\overline{\alpha}}{1-\overline{\alpha}}+\frac{1+z}{1-z}\right)}]\, dr K_\alpha(z) \\
&= \frac{1}{2}\frac{1}{1-\overline{\alpha}z}\left(\frac{1+\overline{\alpha}}{1-\overline{\alpha}}+\frac{1+z}{1-z}\right)\int_0^1 r^{\frac{1}{2}\left(2t+\frac{1+\overline{\alpha}}{1-\overline{\alpha}}+\frac{1+z}{1-z}\right)-1}\, dr \\
&= \frac{1}{1-\overline{\alpha}z}\left(\frac{1+\overline{\alpha}}{1-\overline{\alpha}}+\frac{1+z}{1-z}\right)\frac{1}{2t+\frac{1+\overline{\alpha}}{1-\overline{\alpha}}+\frac{1+z}{1-z}} \\
&= \frac{1}{1+t-tz}\frac{1}{1-\overline{\alpha}\frac{(1-t)z+t}{1+t-tz}} \\
&= T_{f_t}C_{\varphi_t}K_\alpha \\
&= A_tK_\alpha
\end{aligned}$$

Since the span of the kernel functions is dense in H^2, and the integral represents a bounded operator, the equality holds for all vectors in H^2 and the theorem is proved. □

6. Applications of the semigroups in the continuous case

In this section, we will extend the results on semigroups discussed in the previous section and use these and some of the results of the previous section to find the polar decomposition, the absolute value, and the Aluthge transform of some composition operators on H^2. Recall that a bounded linear operator T on Hilbert space has a unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{1/2}$ and U is the appropriate partial isometry. Associated with T, there is a very useful related operator $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$, called the *Aluthge transform of* T (see [12] for more details).

First, we use ideas about coboundaries to give an extended version of Corollary 11 and this provides a different proof of the continuity of the semigroup which is part of that result. We follow the terminology and definitions from Siskakis [16, 17] and Konig [13]. A *coboundary* for $\{\varphi_t : t \geq 0\}$ is a cocycle that can be written in the form

$$\{f_t(z) = \frac{w(\varphi_t(z))}{w(z)} : t \geq 0\}$$

for all z in $\mathbb{D}$ and a suitable analytic function w.

Theorem 26. *For $0 \leq t < \infty$, the cocycle f_t in the definition of the operator A_t is a coboundary with $w(z) = z - 1$. Therefore, A_t is a strongly continuous semigroup of Hermitian weighted composition operators.*

Proof. We have seen in the previous section that A_t is a semigroup of Hermitian weighted composition operators.

For $w(z) = z - 1$,

$$\frac{w(\varphi_t(z))}{w(z)} = \frac{1 - \frac{t+(1-t)z}{1+t-tz}}{1-z} = \frac{1+t-tz-t-(1-t)z}{(1+t-tz)(1-z)} = \frac{1-z}{(1+t-tz)(1-z)}$$

$$= \frac{1}{1+t-tz} = f_t(z)$$

We observe that for $0 \le t < \infty$,

$$\left\| \frac{w \circ \varphi_t}{w} \right\|_\infty = \|f_t\|_\infty = 1$$

This means that

$$\limsup_{t \to 0} \left\| \frac{w \circ \varphi_t}{w} \right\|_\infty = 1$$

and this implies [17, p. 245] that A_t is strongly continuous on H^2. □

These ideas about the semigroups above can be used to find the polar decomposition, the absolute value, and the Aluthge transform of some composition operators on H^2.

To compute an example, we let $\sigma_s(z) = e^{-s}z + 1 - e^{-s}$ for $s \ge 0$ and z in the unit disk. This is a continuous semigroup of maps of the unit disk into itself, and it is known that for each positive number s, the operator $C^*_{\sigma_s}$ on H^2 is a subnormal operator whose spectrum is the disk $\{\lambda : |\lambda| \le e^{s/2}\}$.

Theorem 27. *For any positive numbers p and s*

$$e^{ps}A_{pt} = (C^*_{\sigma_s} C_{\sigma_s})^p$$

where $t = e^s - 1$.

Proof. First, we note [7, p. 322] or [5, Thm. 2], that $C^*_{\sigma_s} = T_g C_\psi T^*_h$ where

$$g(z) = \frac{1}{-(1-e^{-s})z+1}, \qquad h(z) = 0z + 1, \qquad \text{and} \quad \psi(z) = \frac{e^{-s}z}{-(1-e^{-s})z+1}$$

Since $T^*_h = I$, it follows that

$$C^*_{\sigma_s} C_{\sigma_s} = T_g C_\psi C_{\sigma_s} = T_g C_{\sigma_s \circ \psi}$$

Calculating $\sigma_s \circ \psi$ and rewriting with the goal of connecting this to the notation of the previous section, we have

$$\sigma_s \circ \psi = e^{-s}\frac{e^{-s}z}{-(1-e^{-s})z+1} + 1 - e^{-s} = \frac{e^{-2s}z - (1-e^{-s})^2 z + 1 - e^{-s}}{-(1-e^{-s})z+1}$$

$$= \frac{(-1+2e^{-s})z + 1 - e^{-s}}{-(1-e^{-s})z+1} = \frac{(-e^s+2)z + e^s - 1}{-(e^s-1)z + e^s}$$

$$= \frac{(1-t)z+t}{-tz+t+1} = \varphi_t(z)$$

for $t = e^s - 1$. Similarly, we have

$$g(z) = \frac{1}{-(1-e^{-s})z+1} = \frac{e^s}{-(e^s-1)z+e^s} = \frac{e^s}{-tz+1+t} = e^s f_t$$

Thus, we have

$$C_{\sigma_s}^* C_{\sigma_s} = T_g C_{\sigma_s \circ \psi} = e^s T_{f_t} C_{\varphi_t} = e^s A_t$$

The semigroup property for A_t shows $(A_t)^p = A_{pt}$, so the result follows. □

Corollary 28. *For any positive number s, the absolute value of C_{σ_s} is given by*

$$|C_{\sigma_s}| = e^{s/2} A_{t/2}$$

where $t = e^s - 1$.

Proof. This is the case $p = 1/2$ in Theorem 27. □

We want to use the calculation of the absolute value to get the polar decomposition, that is, we want to find a unitary operator U so that $C_{\sigma_s} = U|C_{\sigma_s}|$. If $|C_{\sigma_s}|$ were invertible, then $U = C_{\sigma_s}|C_{\sigma_s}|^{-1}$. In this case, of course, $|C_{\sigma_s}|$ is *not* invertible, but we will proceed, formally, anyway. Corollary 28 says $|C_{\sigma_s}| = TC$ where T is some analytic Toeplitz operator and C is some composition operator. If it made sense, this would mean that $U = C_{\sigma_s}(TC)^{-1} = C_{\sigma_s} C^{-1} T^{-1}$. Of course, formally at least, C^{-1} is another composition operator and T^{-1} is another analytic Toeplitz operator. These considerations motivate the statement of the following result, which is easy to prove once the correct statement has been discovered.

Theorem 29. *For any positive number s, the polar decomposition of C_{σ_s} is $C_{\sigma_s} = U_s|C_{\sigma_s}|$ where $|C_{\sigma_s}| = e^{s/2}A_{t/2}$ and the unitary operator is $U_s = C_{\zeta_s} T_{k_s}$ for*

$$k_s(z) = \frac{(1-e^s)z + e^s + 1}{2e^{s/2}} \qquad \textit{and} \qquad \zeta_s(z) = \frac{(e^s+1)z + e^s - 1}{(e^s-1)z + e^s + 1}$$

for $t = e^s - 1$.

Proof. It is clear that T_{k_s} is a bounded analytic Toeplitz operator. The function ζ_s is an automorphism of the disk

$$\zeta_s(z) = \frac{(e^s+1)z + e^s - 1}{(e^s-1)z + e^s + 1} = \frac{z + \frac{e^s-1}{e^s+1}}{\frac{e^s-1}{e^s+1}z + 1} \tag{12}$$

because $-1 < \frac{e^s-1}{e^s+1} < 1$ for s positive. This means that C_{ζ_s} is bounded on H^2 and U is bounded operator.

From Corollary 28 and the equality $t = e^s - 1$, for any function f in H^2 and any z in the disk, we have

$$\begin{aligned}(|C_{\sigma_s}|f)(z) &= \left(e^{s/2}A_{t/2}f\right)(z) = e^{s/2} f_{t/2}(z)\left(C_{\varphi_{t/2}}f\right)(z) \\ &= \frac{e^{s/2}}{-\frac{t}{2}z + \frac{t}{2} + 1}\left(C_{\varphi_{t/2}}f\right)(z) = \frac{2e^{s/2}}{(1-e^s)z + e^s + 1}\left(C_{\varphi_{t/2}}f\right)(z)\end{aligned}$$

Because the multiplier in the above expression for $(|C_{\sigma_s}|f)(z)$ is just the reciprocal of k_s, we see easily that

$$U_s|C_{\sigma_s}| = C_{\zeta_s} C_{\varphi_{t/2}} = C_{\varphi_{t/2} \circ \zeta_s}$$

The symbol for the latter composition operator is

$$\begin{aligned}
\varphi_{t/2}(\zeta_s(z)) &= \frac{\frac{t}{2}+(1-\frac{t}{2})\zeta_s(z)}{1+\frac{t}{2}-\frac{t}{2}\zeta_s(z)} = \frac{t+(2-t)\zeta_s(z)}{2+t-t\zeta_s(z)} \\
&= \frac{e^s-1+(3-e^s)\zeta_s(z)}{1+e^s-(e^s-1)\zeta_s(z)} = \frac{e^s-1+(3-e^s)\frac{(e^s+1)z+e^s-1}{(e^s-1)z+e^s+1}}{1+e^s-(e^s-1)\frac{(e^s+1)z+e^s-1}{(e^s-1)z+e^s+1}} \\
&= \frac{(e^{2s}-2e^s+1)z+e^{2s}-1+(3+2e^s-e^{2s})z+4e^s-3-e^{2s}}{(e^{2s}-1)z+e^{2s}+2e^s+1-(e^{2s}-1)z-e^{2s}+2e^s-1} \\
&= \frac{4z+4e^s-4}{4e^s} = e^{-s}z+1-e^{-s} = \sigma_s(z)
\end{aligned}$$

That is, for this choice of U_s, we have $U_s|C_{\sigma_s}| = C_{\sigma_s}$.

Finally, we can rewrite $U_s = C_{\zeta_s}T_{k_s}$ to understand it better. Letting

$$a = \frac{e^s-1}{e^s+1} \qquad \text{and} \qquad \xi(z) = \frac{z-a}{az-1}$$

we see from Equation (12) above that

$$\zeta_s(z) = \frac{z+a}{az+1} = \frac{-z-a}{-az-1} = \frac{(-z)-a}{a(-z)-1} = \xi(-z)$$

so that $C_{\zeta_s} = C_{-z}C_\xi$. Furthermore, we see that, for h in H^2,

$$\left(C_\xi T_{k_s}h\right)(z) = \left(C_\xi k_s h\right)(z) = k_s(\xi(z))h(\xi(z)) = \left(T_f C_\xi h\right)(x)$$

where

$$\begin{aligned}
f(z) &= k_s(\xi(z)) = \frac{(1-e^s)\frac{z-\frac{e^s-1}{e^s+1}}{\frac{e^s-1}{e^s+1}z-1}+e^s+1}{2e^{s/2}} = \frac{(1-e^s)\frac{(e^s+1)z-(e^s-1)}{(e^s-1)z-(e^s+1)}+e^s+1}{2e^{s/2}} \\
&= \frac{(1-e^s)((e^s+1)z-(e^s-1))+(e^s+1)((e^s-1)z-(e^s+1))}{2e^{s/2}((e^s-1)z-(e^s+1))} \\
&= \frac{(1-e^{2s}+e2s-1)z+e^{2s}-2e^s+1-e^{2s}-2e^s-1}{2e^{s/2}((e^s-1)z-(e^s+1))} \\
&= \frac{-4e^s}{2e^{s/2}((e^s-1)z-(e^s+1))} = \frac{\frac{2e^{s/2}}{e^s+1}}{1-\frac{e^s-1}{e^s+1}z} = \frac{\sqrt{1-a^2}}{1-az}
\end{aligned}$$

In other words, $U_s = C_{-z}\left(T_f C_\xi\right)$ and, from the description in Section 3, both C_{-z} and T_fC_ξ are Hermitian isometric weighted composition operators. That is, U_s is the product of two unitary operators and is therefore unitary also.

We conclude that $C_{\sigma_s} = U_s|C_{\sigma_s}|$ is the polar decomposition of C_{σ_s}, as we were to prove. ∎

Corollary 30. *For any positive number s, the Aluthge transform of C_{σ_s} is given by*

$$\widetilde{C_{\sigma_s}} = |C_{\sigma_s}|^{1/2} U_s |C_{\sigma_s}|^{1/2} = \left(e^{s/4} A_{t/4}\right) U_s \left(e^{s/4} A_{t/4}\right)$$

where $U_s = C_{\zeta_s} T_{k_s}$ and $t = e^s - 1$.

Proof. Corollary 28 and the semigroup properties of the A_t imply

$$|C_{\sigma_s}|^{1/2} = e^{s/4} A_{t/4}$$

□

If it is desired, since each of the factors in the Aluthge transform of C_{σ_s} is the product of an analytic Toeplitz operator and a composition operator, to write $\widetilde{C_{\sigma_s}}$ as a product of an analytic Toeplitz operator and a composition operator as well.

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