A NEW CLASS OF OPERATORS AND A DESCRIPTION OF ADJOINTS OF COMPOSITION OPERATORS

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ABSTRACT. Starting with a general formula, precise but difficult to use, for the adjoint of a composition operator on a functional Hilbert space, we compute an explicit formula on the classical Hardy Hilbert space for the adjoint of a composition operator with rational symbol. To provide a foundation for this formula, we study an extension to the definitions of composition, weighted composition, and Toeplitz operators to include symbols that are multiple valued functions. These definitions can be made on any Banach space of analytic functions on a plane domain, but in this work, our attention is focused on the basic properties needed for the application to operators on the standard Hardy and Bergman Hilbert spaces on the unit disk.

1. INTRODUCTION AND PRELIMINARIES

In the study of operator theory, multiplication or Toeplitz operators, composition operators, and weighted composition operators have played an important role as basic examples and motivators of the theory. The definitions of these operators have used functions in typical ways that incorporate the arithmetic and algebra of the interactions of functions with each other. While it has seemed natural to use single valued functions, in this note, we show that it is not necessary to do so.

The goal of this paper is to introduce composition operators and weighted composition operators on Banach spaces of analytic functions with multiple valued symbols and to describe some of their basic properties such as information about boundedness on some of the most common spaces, including the Hardy and Bergman spaces on the unit disk in the complex plane. Using this definition, we compute the adjoint of composition operators on $H^2(\mathbb{D})$ with rational symbol. In retrospect, it can be seen that the description of the commutants of analytic Toeplitz operators whose symbols are finite Blaschke products as given by Cowen [1, section 2] are multiple valued weighted composition operators.

In addition, we consider some similar constructions in which the resulting functions are not analytic on the expected domain but rather require projection into

a subspace of analytic functions to complete the definition. In this situation, we generalize the notion of Toeplitz operator.

If $\mathcal{H}$ is a functional Hilbert space on a plane domain Ω and φ is an analytic self-map of Ω, then for f in $\mathcal{H}$

$$C_\varphi f = f \circ \varphi$$

defines a composition operator. Although boundedness, compactness, and other properties have been characterized for composition operators in many contexts, other interesting and seemingly basic problems remain open. The computation of adjoints of composition operators is one of these problems.

There have been two exceptions to this statement. In the case that φ is a linear fractional map mapping the unit disk into itself, Cowen [2] showed that the adjoint C_φ^* can be expressed as a product of Toeplitz operators and a composition operator (see also [3, chapter 9]) and this computation has been extended to many other spaces in one and several variables. Second, when φ (not the identity function) is an inner function with $\varphi(0) = 0$, since C_φ is an isometry of infinite multiplicity on $H^2(\mathbb{D})$, the adjoint is easily computed. This computation can be extended to the case when φ has a fixed point in the open disk because then C_φ is similar to such an isometry.

In this paper, we begin with a general formula, that might be considered mathematical folklore, for the adjoint of a composition operator that is usually difficult to use effectively even though in many cases it can be expressed as an integral operator (see also Sarason [7]). The expression was noted in [3, p. 322] for the case of C_φ acting on $H^2(\mathbb{D})$ having a symbol that is an inner function. This idea is developed in Section 2 to provide a formula for the adjoint of a composition operator acting on some Hilbert spaces of analytic functions on $\mathbb{D}$, such as $H^2(\mathbb{D})$ and weighted Bergman spaces.

Recently, Wahl [9] used this kind of formula to provide a formal computation of the adjoint C_φ^* for the multivalent function $\varphi(z) = (1 - 2c)z^2/(1 - 2cz^2)$ with $0 < c < 1/2$, and from this derived other interesting properties of these composition operators. In Section 3 we compute, in a way similar to Wahl, the adjoint on $H^2(\mathbb{D})$ for C_φ where $\varphi(z) = (z^2 + z)/2$ and this computation is used to motivate the definition of multiple valued weighted composition operators. Finally, we prove basic properties of these operators and show that when φ is a rational function, the adjoint of C_φ on the Hardy Hilbert space $H^2(\mathbb{D})$ is a multiple valued weighted composition operator.

Finally, in Section 4, we consider some similar constructions in which the resulting functions are not analytic on the expected domain but rather require projection into a subspace of analytic functions to complete the definition. In this situation, we generalize the notion of Toeplitz operator.

We begin by recalling the definition of a *functional Banach space*, which will be, mainly, the setting of our work.

Definition 1.1. [3, page 2] *A Banach space of complex valued functions on a set Ω is called a* functional Banach space *on Ω if the vector operations are the pointwise operations, $f(x) = g(x)$ for each x in Ω implies $f = g$, $f(x) = f(y)$ for each function in the space implies $x = y$, and for each x in Ω, the linear functional $f \mapsto f(x)$ is continuous.*

When Ω is a domain in the complex plane, a functional Banach space whose functions are analytic on Ω will usually be called a *Banach space of analytic functions.* If $\mathcal{H}$ is a Hilbert space of analytic functions, the final property in the definition above guarantees the existence of vectors in $\mathcal{H}$, often called *reproducing kernel functions*, that give the value of functions f in $\mathcal{H}$ at points x in Ω by taking inner products:

$$\langle f, K_x \rangle = f(x)$$

As is well known, in any Hilbert space of analytic functions, if φ is an analytic map of Ω into itself, this leads to an easy formula for the adjoint of a composition operator acting on a reproducing kernel function. For any f in the space,

(1) $$\langle f, C_\varphi^* K_x \rangle = \langle C_\varphi f, K_x \rangle = \langle f \circ \varphi, K_x \rangle = f(\varphi(x)) = \langle f, K_{\varphi(x)} \rangle$$

Since this is true for any f in the space, $C_\varphi^* K_x = K_{\varphi(x)}$.

A similar idea gives a simple formula for the value at a point of the function $C_\varphi^* f$ for f in the space:

$$(C_\varphi^* f)(x) = \langle C_\varphi^* f, K_x \rangle = \langle f, C_\varphi K_x \rangle = \langle f, K_x \circ \varphi \rangle$$

In the context in which the inner product is given by an integral, a very common situation, this formula becomes an expression for the adjoint of a composition operator as an integral operator. This idea becomes the basis for the results of the next section.

2. A FORMULA FOR THE ADJOINT IN HARDY AND WEIGHTED BERGMAN SPACES

An explicit expression for the adjoint of a composition operator induced by any analytic function φ mapping the unit disk into itself in the Hardy space can be easily computed. In fact, the underlying idea is, as above, simple and it has to do with reproducing kernels.

Let $K_w(z)$ denote *the reproducing kernel at w in $\mathbb{D}$ for $H^2(\mathbb{D})$*, which is given by

$$K_w(z) = \frac{1}{1 - \overline{w}z}, \qquad (z \in \mathbb{D})$$

It is well known (see [4], for instance) that for any two Hardy functions f and g, the inner product in $H^2(\mathbb{D})$, denoted by $\langle f, g \rangle_{H^2(\mathbb{D})}$, can be computed by means of

an integral of their boundary values

$$\langle f,\, g \rangle_{H^2(\mathbb{D})} = \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})}\, \frac{d\theta}{2\pi}$$

Now, if f is in $H^2(\mathbb{D})$, it follows

$$C_\varphi^* f(z) = \langle C_\varphi^* f, K_z \rangle_{H^2(\mathbb{D})} = \langle f, C_\varphi K_z \rangle_{H^2(\mathbb{D})} = \int_0^{2\pi} \frac{f(e^{i\theta})}{1 - \overline{\varphi(e^{i\theta})}z}\, \frac{d\theta}{2\pi}$$

Therefore, we easily get

A formula for C_φ^* on the Hardy space. *Let φ be an analytic function on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. Then the adjoint of the composition operator C_φ on $H^2(\mathbb{D})$ is given by*

$$(2) \qquad (C_\varphi^* f)(z) = \int_0^{2\pi} \frac{f(e^{i\theta})}{1 - \overline{\varphi(e^{i\theta})}z}\, \frac{d\theta}{2\pi}, \qquad (f \in H^2(\mathbb{D}))$$

Observe that no extra hypotheses on φ have been necessary.

Next, we deal with the formula for the adjoint of a composition operator on weighted Bergman spaces. Of course, the underlying idea is the same as in the Hardy space case.

For $\alpha > -1$, recall that the weighted Bergman space $\mathcal{A}_\alpha^2$ consists of analytic functions f on $\mathbb{D}$ for which the norm

$$\|f\|_\alpha^2 = \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\alpha\, dA(z)$$

is finite. Here, $dA(z) = \dfrac{1}{\pi}\, dx\, dy$ denotes the normalized Lebesgue area measure on $\mathbb{D}$. Observe that $\mathcal{A}_\alpha^2$ are Hilbert spaces of analytic functions on $\mathbb{D}$. In addition, if $\alpha = 0$, $\mathcal{A}_\alpha^2$ turns out to be the classical Bergman space $\mathcal{A}^2$.

A little computation shows that the reproducing kernel functions in $\mathcal{A}_\alpha^2$ are

$$K_w(z) = \frac{\alpha + 1}{(1 - \overline{w}z)^{\alpha+2}}, \qquad (w \in \mathbb{D})$$

Therefore, the same argument as before yields

A formula for C_φ^* on weighted Bergman spaces. *Let φ be an analytic function on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. Then the adjoint of the composition operator C_φ on $\mathcal{A}_\alpha^2$ is given by*

$$(3) \qquad (C_\varphi^* f)(z) = \int_{\mathbb{D}} \frac{(\alpha+1)f(w)}{(1 - \overline{\varphi(w)}z)^{\alpha+2}}\, (1 - |w|^2)^\alpha\, dA(w),$$

where $f \in \mathcal{A}_\alpha^2$.

In particular, we may deduce C_φ^* acting on the orthogonal system $\{z^n\}_{n=0}^\infty$ in $\mathcal{A}_\alpha^2$. If D_w denotes the differential operator respect to the variable w, D_w^0 the identity operator, and D_w^n the n-th power of D_w, a little computation shows that $\|z^n\|_\alpha^2 = n!/(\alpha+1)(\alpha+2)\cdots(\alpha+n+1)$ (see also [3, Chapter 3]). Therefore,

Corollary 2.1. *Let φ be a analytic self-map of the unit disk. If $f_n(z) = z^n$, $n \geq 0$, then in the weighted Bergman space $\mathcal{A}_\alpha^2$ the following holds*

$$(C_\varphi^* f_n)(z) = \frac{1}{(\alpha+2)\cdots(\alpha+n+1)} D_w^n \left(\frac{1}{(1 - \overline{\varphi(\overline{w})}z)^{\alpha+2}} \right) \Big|_{w=0}$$

2.1. Weighted Dirichlet spaces.

For $\alpha > -1$, we recall that the weighted Dirichlet space $\mathcal{D}_\alpha$ is the collection of analytic functions f on $\mathbb{D}$ for which the complex derivative f' belongs to the weighted Bergman space $\mathcal{A}_\alpha^2$. Different norms, although all of them equivalent, turn out the weighted Dirichlet spaces in Hilbert spaces of analytic functions. Since the adjoint of an operator in a Hilbert space depends strongly on the norm considered, one can deduce the corresponding expression for C_φ^* once the choice of the norm is accomplished and the reproducing kernels K_w are determined. In particular, this is the case with the formula obtained in [5] for the adjoint of linear fractional composition operators on the Dirichet space modulo constant functions.

3. The Analytic Case: Multiple valued weighted composition operator

In this section, we consider the generalization to the multiple valued case of composition operators on spaces of analytic functions and analytic Toeplitz operators which are, of course, just multiplication operators. Basic properties of such operators acting on functional Banach spaces of analytic functions are also studied.

In particular, we will describe the adjoint of a composition operator induced by a rational function acting on the Hardy space as a multiple valued composition operator. We begin by considering an example which sheds some light on the spirit of the whole section.

3.1. An example.

Let $\varphi(z) = (z^2 + z)/2$, with z in $\mathbb{D}$. It is clear that φ takes $\mathbb{D}$ into itself and induces a bounded composition operator on $H^2(\mathbb{D})$. We have

$$
\begin{aligned}
C_\varphi^* f(z) &= \frac{1}{\pi} \int_0^{2\pi} \frac{f(e^{i\theta})}{2 - (e^{-2i\theta} + e^{-i\theta})z} \, d\theta \\
&= \frac{1}{\pi} \int_0^{2\pi} \frac{e^{2i\theta} f(e^{i\theta})}{2e^{2i\theta} - (1 + e^{i\theta})z} \, d\theta \\
&= \frac{1}{\pi i} \int_{\partial \mathbb{D}} \frac{\zeta f(\zeta)}{2\zeta^2 - (1 + \zeta)z} \, d\zeta,
\end{aligned}
$$

(4)

where in last line the change of variable $\zeta = e^{i\theta}$ has been carried out. Now, observe that $2\zeta^2 - (1 + \zeta)z = 2(\zeta - \zeta_1)(\zeta - \zeta_2)$, where

$$\zeta_1 = \frac{z + \sqrt{z^2 + 8z}}{4} \qquad \text{and} \qquad \zeta_2 = \frac{z - \sqrt{z^2 + 8z}}{4}.$$

If $z \neq 0$, upon applying the Residue's Theorem, it follows that the integral in (4) is equal to

$$\frac{1}{\zeta_1 - \zeta_2} \left(\zeta_1 f(\zeta_1) - \zeta_2 f(\zeta_2) \right)$$

or equivalently, we have

$$C_\varphi^* f(z) = \frac{z + \sqrt{z^2 + 8z}}{2\sqrt{z^2 + 8z}} \, f\left(\frac{z + \sqrt{z^2 + 8z}}{4} \right) - \frac{z - \sqrt{z^2 + 8z}}{2\sqrt{z^2 + 8z}} \, f\left(\frac{z - \sqrt{z^2 + 8z}}{4} \right)$$

for z in the disk and different from zero. This means, formally, that we may express C_φ^* by

$$(5) \qquad\qquad C_\varphi^* f(z) = \sum \psi(z)(f \circ \sigma)(z),$$

where $\psi(z) = \left(z \pm \sqrt{z^2 + 8z} \right)/2\sqrt{z^2 + 8z}$, $\sigma(z) = (z \pm \sqrt{z^2 + 8z})/4$ and the sum is taken over all the branches of ψ and σ combined, roughly speaking, in a special way. The important point to note here is that we are not allowed to combine all the branches of ψ and σ. Although Equation (5) is a formal expression, the principal significance of this example is the relationship between the adjoint of a composition operator and certain weighted composition operators. In what follows, we formalize such a relation.

3.2. Multiple Valued Composition Operators.

Definition 3.1. *Suppose Ω is a domain in the complex plane and b is a point of Ω, the* base point. *Let K be a finite set in Ω that does not include b, suppose ψ and σ are functions analytic in a simply-connected neighborhood of b in $\Omega \setminus K$ and suppose they are arbitrarily continuable in $\Omega \setminus K$. We say (ψ, σ) is a* compatible *pair of multiple valued (analytic) functions on Ω if for any path γ in $\Omega \setminus K$ along which the continuation of σ yields the same branch as at the beginning, it is also the case that continuation of ψ along γ yields the same branch as at the beginning.*

The definition above apparently depends on the base point b and the finite set K and on the functions ψ and σ defined in a neighborhood of b identified in the statement. However, informally, we regard ψ and σ as the names of the multiple valued functions defined in $\Omega \setminus K$ and the intent of the definition is to connect each branch of σ at a point with a specific branch of ψ at that point; the initial branches of ψ and σ defined in the simply connected neighborhood of b gives one pairing of the branches and continuation of these gives all other pairings of the branches that can occur in this context. Because this association is independent of the base point b and because the smallest allowable K for given Ω and σ follow from the definition of σ, we do not usually include the base point and the set K in the description, but just say (ψ, σ) is a compatible pair of multiple valued functions on Ω, with the specific pairing of their branches as part of the definition of the compatible pair. Note that it is a consequence of the definition that if (ψ, σ) is a compatible pair

on Ω then the number of branches of ψ at a point is a divisor of the number of branches of σ at that point (provided that either is finite). The number of branches of σ at each point will be called the *cardinality of the pair* or sometimes the pair of integers (number of branches of ψ, number of branches of σ) will be called the cardinality of the pair. Also note that if σ has a removable singularity at a point z_0 in K and if ψ is bounded in a punctured neighborhood of z_0, then ψ has a removable singularity at z_0 as well: because each branch of σ is single valued in a neighborhood of z_0, the fact that each branch of σ is associated with a particular branch of ψ means that ψ is also single valued in a neighborhood of z_0.

For example, suppose Ω is the unit disk, $K = \{0\}$, and suppose $b = 1/16$ has been chosen as the base point. Let $\sigma(z) = z^{1/4}$ with $\sigma(b) = 1/2$, let $\psi_1(z) = z^{1/2}$ with $\psi_1(b) = 1/4$, and let $\psi_2(z) = z^{1/2}$ with $\psi_2(b) = -1/4$. Then (ψ_1, σ) and $\psi_2, \sigma)$ are both compatible pairings on the disk, but they are different from each other because pairings of the branches are different. Note also that if $\psi(z) = z^3$ with $\psi(b) = 1/4096$, we also have (ψ, σ) as a compatible pairing, although perhaps a less interesting compatible pairing than the other two examples.

Definition 3.2. *Suppose Ω is a domain in the complex plane, suppose K is a finite subset of Ω and suppose $\mathcal{H}$ is a Banach space of analytic functions on Ω. Suppose, further, that ψ and σ are analytic functions that are arbitrarily continuable in $\Omega \setminus K$ and that (ψ, σ) is a compatible pair with finite multiplicity. If σ maps $\Omega \setminus K$ into Ω such that, for each branch σ_j of σ, the cluster set of σ_j for z near b in K (for each b in K) does not intersect the boundary of Ω, and $\lim_{z \to b} \psi_j(z)(z-b) = 0$ for all b in K and each branch ψ_j, the* multiple valued weighted composition operator *$W_{\psi, \sigma}$ on $\mathcal{H}$ is the operator defined by*

$$W_{\psi, \sigma} f(z) = \sum \psi(z) f(\sigma(z))$$

for f in $\mathcal{H}$ where the sum is taken over all the branches of the pair (ψ, σ), and z in $\Omega \setminus K$.

Note: In the balance of this paper, we will write $\lim_{z \to b} \psi(z)(z-b) = 0$ to mean $\lim_{z \to b} \psi_j(z)(z-b) = 0$ for all branches ψ_j of ψ, and similar shorthand, rather than emphasize the separate branches of ψ or σ.

Before going further, we point out that the formula $W_{\psi, \sigma} f(z) = \sum \psi(z) f(\sigma(z))$ defines an analytic function in Ω for each f analytic on Ω. Indeed, because σ and ψ are arbitrarily continuable in $\Omega \setminus K$ and because σ maps $\Omega \setminus K$ into Ω, included in the domain of f, the formula makes sense in a neighborhood of each point of $\Omega \setminus K$ and is arbitrarily continuable in $\Omega \setminus K$. Moreover, because ψ and σ form a compatible pair, and each branch of σ occurs in the sum exactly once, continuing the locally defined terms of the sum only permutes their order and the sum is single

valued. Finally, for each b in K,

$$\lim_{z \to b}(z - b)\left(\sum \psi(z)f(\sigma(z))\right) = \sum \lim_{z \to b}(z - b)\psi(z)f(\sigma(z))$$

Now, for each branch of ψ, we have $\lim_{z \to b}(z - b)\psi(z) = 0$. Since σ maps $\Omega \setminus K$ into Ω and the cluster set of each branch of σ as z approaches b does not intersect the boundary of Ω, we have $|f(\sigma(z))|$ is bounded and z approaches b, so we have $\lim_{z \to b}(z - b)\psi(z)f(\sigma(z)) = 0$ for each branch of σ. Since

$$\lim_{z \to b}(z - b)\left(\sum \psi(z)f(\sigma(z))\right) = 0$$

for each b in K, each point of K is a removable singularity of $\sum \psi(z)f(\sigma(z))$ and $W_{\psi,\sigma}f$ is well defined.

Now, we proceed to define multiple valued composition operators. Note that if σ is a multiple valued analytic function on $\Omega \setminus K$ that maps $\Omega \setminus K$ into Ω and ψ is constant function 1, then (ψ, σ) is a compatible pair on Ω.

Definition 3.3. *Suppose Ω is a domain in the complex plane, suppose K is a finite subset of Ω, and suppose $\mathcal{H}$ is a Banach space of analytic functions on Ω. Suppose, further, that σ is an analytic function as in Definition 3.2 that is arbitrarily continuable in $\Omega \setminus K$, takes values in Ω, and has finite multiplicity. The multiple valued composition operator C_σ on $\mathcal{H}$ is $C_\sigma = W_{1,\sigma}$, that is, it is the operator defined by*

$$C_\sigma f(z) = \sum f(\sigma(z))$$

for f in $\mathcal{H}$, where the sum is taken over all the branches of σ.

Remark 3.4. Note that the restriction to finite multiplicity of σ in the above definition (and also in Definition 3.2) is sufficient to guarantee convergence of the series in the definition of the operators. In fact, if we consider the inner function $\varphi(z) = \exp(z + 1)/(z - 1)$ defined on $\mathbb{D}$ and take σ to be the multiple valued map φ^{-1}, it follows that the series $\sum \sigma(z)$, taken over all the branches of σ, diverges for all z in $\mathbb{D}$, except on a set of logarithmic capacity zero.

On the other hand, it might seem that the condition on compatibility of σ and ψ is too restrictive. For example, one might contemplate extending this pair of definitions to include multiple valued Toeplitz operators or to pairs (ψ, σ) where the number of branches of ψ is greater than the number of branches of σ. This turns out to be unnecessary. For example, suppose $\sigma(z) = z$ and $\psi(z) = \sqrt{z}$ and we consider the multiple valued weighted composition operator on $H^2(\mathbb{D})$, or better, the multiple valued Toeplitz operator on $H^2(\mathbb{D})$ given by

$$T_\psi f(z) = \left(\sqrt{z}f(\sigma(z)) + (-\sqrt{z})f(\sigma(z))\right) = \left(\sqrt{z}f(z) + (-\sqrt{z})f(z)\right)$$

as defined above. Then

$$T_\psi f(z) = \left(\sqrt{z} + (-\sqrt{z})\right)f(z) = 0$$

More generally, if ψ is any multiple valued function on $\Omega \setminus K$ with finite multiplicity n and $\sigma(z) = z$, the analogy to the above definition would give

$$W_{\psi,\sigma} f(z) = \sum \psi(z) f(\sigma(z)) = \left(\sum \psi(z) \right) f(z)$$

But since $\sum \psi(z)$ is arbitrarily continuable in $\Omega \setminus K$ and since each term in the sum continues to a different branch also in the original sum, the summands are just permuted in the sum and the total is the same. That is, the function $\sum \psi(z)$ is arbitrarily continuable and single valued in $\Omega \setminus K$ and since $\lim_{z \to b} (z - b) \sum \psi(z) = 0$ for each b in K, this means that $\sum \psi(z)$ is analytic in Ω. Thus, the "multiple valued Toeplitz operator" intended to be defined is really just an ordinary single valued Toeplitz operator.

Now, we prove the following result on boundedness of these operators.

Theorem 3.5. *If $W_{\psi,\sigma}$ is a multiple valued weighted composition operator on $H^2(\mathbb{D})$ such that ψ and σ satisfy the conditions of Definition 3.2. Assume also that $M = \limsup_{|z| \to 1^-} |\psi(z)| < \infty$. Then $W_{\psi,\sigma}$ is a bounded operator on $H^2(\mathbb{D})$ and*

$$\|W_{\psi,\sigma}\| \leq M \sqrt{n \sum \frac{1 + |\sigma(0)|}{1 - |\sigma(0)|}},$$

where the latter sum is taken over the branches of σ, and n is the multiplicity of the compatible pair (ψ, σ).

Proof. Let R be a positive number such that $R < 1$ and $R > \max_{b \in K} |b|$. Let $M_R = \sup_{R < |z| < 1} |\psi(z)|$. For such R, we see that $M_R < \infty$ because ψ is analytic at z for $R < |z| < 1$, continuous for $|z| = R$, and $\limsup_{|z| \to 1^-} |\psi(z)| = M < \infty$. In fact, $\lim_{R \to 1^-} M_R = M$.

Suppose f is in $H^2(\mathbb{D})$, and suppose h_f is the least harmonic majorant for $|f|^2$, which means that $h_f(0) = \|f\|^2$ (See [4, Chap. ?] for the connection between the Hardy space and harmonic majorants). Then denoting by $\tilde{h}_f$ the harmonic conjugate of h_f with value 0 at 0, we have $h_f + i\tilde{h}_f$ is analytic in the open unit disk and

$$\sum h_f \circ \sigma + i\tilde{h}_f \circ \sigma$$

is analytic and single valued in $\mathbb{D} \setminus K$. In particular, then, $\sum h_f \circ \sigma$ is a positive harmonic function in the open unit disk.

If z satisfies $R < |z| < 1$, then

$$
\begin{aligned}
\left| \sum \psi(z) f(\sigma(z)) \right|^2 &\leq \left(\sum |\psi(z)| |f(\sigma(z))| \right)^2 \\
&\leq M_R^2 \left(\sum |f(\sigma(z))| \right)^2 \\
&\leq M_R^2 n \sum |f(\sigma(z))|^2 \\
&\leq M_R^2 n \sum h_f(\sigma(z))
\end{aligned}
$$

Note that the penultimate inequality is obtained by using Cauchy–Schwartz inequality for the inner product of the vector of ones with the vector of values of $|f(\sigma(z))|$. This inequality says that the positive harmonic function $M_R^2 \, n \sum h_f(\sigma(z))$ dominates the subharmonic function $|\sum \psi(z)f(\sigma(z))|^2$ on the annulus $R < |z| < 1$ from which it follows that it dominates on the whole open unit disk.

Applying Harnack's inequality to h_f, we get

$$\sum h_f(\sigma(0)) \leq \sum h_f(0)\frac{1+|\sigma(0)|}{1-|\sigma(0)|} = \left(\sum \frac{1+|\sigma(0)|}{1-|\sigma(0)|} \right) \|f\|^2$$

Combining these inequalities and noting that the M_R's converge to M, we get, for each f in $H^2(\mathbb{D})$,

$$\|W_{\psi,\sigma}f\| \leq M\sqrt{n\sum \frac{1+|\sigma(0)|}{1-|\sigma(0)|}}\|f\|$$

and the desired inequality follows immediately. $\qquad\square$

Corollary 3.6. *If σ is a multiple valued map of multiplicity n of $\mathbb{D} \setminus K$ into $\mathbb{D}$, then C_σ is bounded on $H^2(\mathbb{D})$ and*

$$\|C_\sigma\| \leq \sqrt{n\sum \frac{1+|\sigma(0)|}{1-|\sigma(0)|}}$$

Next result generalizes to multiple valued weighted composition operators the well known result given by Equation 1 for ordinary composition operators.

Theorem 3.7. *Suppose K is a finite set in the domain Ω, (ψ, σ) is a compatible pair of multiple valued functions on $\Omega \setminus K$, and n is the cardinality of the pair. Suppose $W_{\psi,\sigma}$ is a multiple valued weighted composition operator on a Banach space $\mathcal{B}$ of functions analytic on Ω. If K_α is the kernel for evaluation of functions of $\mathcal{B}$ at the point α of Ω, then for each α in $\Omega \setminus K$, there are points β_1, β_2, $\cdots$, β_n of Ω such that $W_{\psi,\sigma}^* K_\alpha$ is a linear combination of the K_{β_j}, that is,*

$$W_{\psi,\sigma}^* K_\alpha = \sum_{j=1}^n a_j K_{\beta_j}$$

Proof. To show that the desired equality, we firstly observe that

$$\langle f, W_{\psi,\sigma}^* K_\alpha \rangle = \langle W_{\psi,\sigma}f, K_\alpha \rangle = \left(\sum \psi f \circ \sigma \right)(\alpha) = \sum \psi(\alpha)f(\sigma(\alpha)) = \sum a_j f(\beta_j)$$

where the points β_1, β_2, $\cdots$, β_n are the images of α under the branches of σ and the numbers a_1, a_2, $\cdots$, a_n are the values at α under the branches of ψ.

That is,

$$\langle f, W_{\psi,\sigma}^* K_\alpha \rangle = \sum a_j \langle f, K_{\beta_j} \rangle$$

and the last sum is either $\langle f, \sum a_j K_{\beta_j} \rangle$ or $\langle f, \sum \overline{a_j} K_{\beta_j} \rangle$ depending on whether $\mathcal{B}$ is a Hilbert space or not. Notice that the numbers a_j and β_j do not depend on f, only on α and the functions ψ and σ. Since the above equality holds for all f in

$\mathcal{B}$, we see that $W^*_{\psi,\sigma} K_\alpha$ is $\sum a_j K_{\beta_j}$ or $\sum \overline{a_j} K_{\beta_j}$, a linear combination of evaluation kernels for at most n points of Ω, as was required. $\qquad \square$

3.3. C^*_φ as a multiple valued weighted composition operator.

In this subsection, we express the adjoint of composition operators induced by rational maps as multiple valued weighted composition operators. In fact, expressions generalize, in some sense, the expression for the adjoint of composition operators induced by linear fractional maps obtained in [2].

Theorem 3.8. *Let φ be a rational map taking $\mathbb{D}$ into itself. Let $\widetilde{\varphi^{-1}}$ denote the multiple valued algebraic function defined by $\widetilde{\varphi^{-1}}(z) = \overline{\varphi^{-1}(1/\overline{z})}$. Then, for any f in $H^2(\mathbb{D})$*

$$C^*_\varphi f(z) = B W_{\psi,\sigma} f(z),$$

where B is the backward shift operator and $W_{\psi,\sigma}$ is the multiple valued weighted composition operator induced by $\sigma = 1/\widetilde{\varphi^{-1}}$ and $\psi = \widetilde{(\varphi^{-1})'}/\widetilde{\varphi^{-1}}$.

Before proceeding further, observe that σ is a multiple valued function that takes $\mathbb{D}$ into itself since the the rational map φ is assumed to do so. In addition, no branches of $\widetilde{\varphi^{-1}}$ vanish in $\mathbb{D}$, and ψ and σ satisfy the properties of Definition 3.2, so $W_{\psi,\sigma}$ is well defined on $\mathbb{D}$.

Proof of Theorem 3.8. In order to get an expression for C^*_φ, let f and g be polynomials. Let $\widetilde{g}$ be the holomorphic function in $\{z : |z| > 1\}$ defined by $\widetilde{g}(z) = \overline{g(1/\overline{z})}$. It holds that if g has non-tangential limit $g(\zeta)$ at ζ in $\partial\mathbb{D}$, then so does $\widetilde{g}$ and $\widetilde{g}(\zeta) = \overline{g(\zeta)}$. Then,

$$
\begin{aligned}
\langle f, C^*_\varphi g \rangle_{H^2(\mathbb{D})} &= \frac{1}{2\pi} \int_0^{2\pi} f(\varphi(e^{i\theta})) \overline{g(e^{i\theta})} \, d\theta \\
&= \int_{\partial\mathbb{D}} f(\varphi(\zeta)) \widetilde{g}(\zeta) \frac{d\zeta}{\zeta}.
\end{aligned}
$$

Let $\varphi^{-1}(\mathbb{D})$ denote the component in $\mathbb{C}$ which contains $\mathbb{D}$. Observe that since φ has no poles in $\varphi^{-1}(\mathbb{D})$, and $\widetilde{g}$ is holomorphic $\{z : |z| > 1\}$, Cauchy's Theorem yields

$$(6) \qquad \langle f, C^*_\varphi g \rangle_{H^2(\mathbb{D})} = \int_{\partial\varphi^{-1}(\mathbb{D})} f(\varphi(\zeta)) \widetilde{g}(\zeta) \frac{d\zeta}{\zeta}.$$

Now, since φ is a rational map, there exists a positive integer N and N arcs of curve $\Gamma_j \subset \partial\varphi^{-1}(\mathbb{D})$ such that

$$\partial\varphi^{-1}(\mathbb{D}) = \bigcup_{j=1}^{N} \Gamma_j,$$

$\Gamma_j \cap \Gamma_k = \emptyset$ for $j \neq k$ and, for each $j = 1, \cdots, N$ the arc of curve Γ_j is mapped onto $\partial\mathbb{D}$ by φ. Thus, the integral in (6) is equal to

$$(7) \qquad \sum_{j=1}^{N} \int_{\Gamma_j} f(\varphi(\zeta)) \widetilde{g}(\zeta) \frac{d\zeta}{\zeta}.$$

At this point, we may proceed with the change of variables $\varphi(\xi) = \eta$ in each of the integrals involved in the sum in (7), since φ takes Γ_j bijectively onto $\partial\mathbb{D}$. Once again, abusing the notation, we will write φ^{-1} for all branches φ_j^{-1} of φ^{-1}. Then, it follows

$$\langle f, C_\varphi^* g\rangle_{H^2(\mathbb{D})} \quad = \int_{\partial\mathbb{D}} \eta\, f(\eta) \left(\sum_{j=1}^{N} \frac{\widetilde{g}(\varphi^{-1}(\eta))}{\varphi^{-1}(\eta)\, \varphi'(\varphi^{-1}(\eta))}\right) \frac{d\eta}{\eta}$$

Now, observe that whenever η is in $\partial\mathbb{D}$ we have

$$\widetilde{g}(\varphi^{-1}(\eta)) = \overline{g\left(\frac{1}{\widetilde{\varphi^{-1}}(\eta)}\right)}.$$

Therefore, a little computation shows that

$$\langle f, C_\varphi^* g\rangle_{H^2(\mathbb{D})} = \int_{\partial\mathbb{D}} \eta\, f(\eta) \overline{\left(\sum \frac{\widetilde{(\varphi^{-1})'}(\eta)}{\widetilde{\varphi^{-1}}(\eta)}\ g\left(\frac{1}{\widetilde{\varphi^{-1}}(\eta)}\right)\right)} \frac{d\eta}{\eta}$$

which is the desired expression.

$\square$

This gives a simple condition for the functions in the kernel of C_φ^* when φ is a rational function.

Corollary 3.9. *Let φ be a rational function taking $\mathbb{D}$ into itself and let ψ and σ be defined as in the statement of Theorem 3.8. Then f in $H^2(\mathbb{D})$ is in the kernel of C_φ^* if and only if*

$$\sum \psi(z)\, f(\sigma(z)) = \sum \psi(0) f(\sigma(0))$$

for all z in $\mathbb{D}$.

Proof. The function f in $H^2(\mathbb{D})$ is in the kernel of C_φ^* if and only if $BW_{\psi,\sigma}f$ is the zero function. Since the kernel of the backward shift operator B is the subspace of constant functions, this means f is in the kernel of C_φ^* if and only if $W_{\psi,\sigma}f$ is a constant function. That is, the value of $\sum \psi(z)\, f(\sigma(z))$ is the same for every z in $\mathbb{D}$ as it is at $z = 0$.

$\square$

4. The Measurable Case: projected multiple valued weighted composition operators

The definitions in Section 3 can be extended to cases in which the multiple valued functions which are symbols for the operator are no longer analytic in the expected domain. In this case, in addition to considering the generalization to multiple valued functions, we must project the function constructed by the operator in L^2 back into the subspace of analytic functions, as in the case of Toeplitz operators, to complete the definition. The purpose of this section is to carry out the study of such operators.

We begin with a Hilbert space of analytic functions $\mathcal{H}$ that is a closed subspace of $L^2(X, d\mu)$ where each function of $\mathcal{H}$ can be extended to a subset of full measure in X. For some positive integer n, let Y be $X \times \{1, 2, 3, \cdots, n\}$ with the product topology and let π be the map $\pi(x, j) = x$ for (x, j) in Y. When Δ is an open subset of Y such that $\pi(\Delta)$ has full measure in X, we say that Δ *almost covers X with finite multiplicity.*

Suppose S is a measurable map of Δ to X and Ψ is a measurable complex valued function on Δ. Alternatively, we may describe this as n maps S_1, S_2, $\cdots$, S_n on X, where for each j, the function $\S_j(x)$ is defined on the open set $\{x : (x, j) \in \Delta \cap X \times \{j\} \}$ by $S_j(x) = S(x, j)$. Similarly, we define functions Ψ_j by $\Psi_j(x) = \Psi(x, j)$.

Definition 4.1. *The* compatible pair (ψ, σ) of multiple valued (measurable) func-tions *on X is the collection of the maps defined above by Y, Δ, Ψ, and S where the 'values' of ψ at a point x of X are the values $\{\Psi_j(x) | (x, j) \in \Delta\}$ and the 'values' of σ at a point x of X are the values $\{S_j(x) | (x, j) \in \Delta\}$.*

Definition 4.2. *Suppose (ψ, σ) is a compatible pair of multiple valued (measurable) functions on X as in the definition above. We say (ψ, σ) is a* simple compatible pair of multiple valued (measurable) functions *on X if each of the functions S_j is one-to-one on its domain $\{x | (x, j) \in \Delta\}$.*

In some sense, to call ψ and σ 'multiple valued functions' is an abuse of ter-minology, but since the very term 'multiple valued function' is self-contradictory, this seems to the authors to be a minor transgression that is covered by the careful definition above of its meaning. With this in mind, we will use terminology like ψ is in L^∞, or σ is C^1, and so on, to mean more precisely Ψ is in $L^\infty(\Delta)$ and S is C^1 on Δ.

Note that the conditions on X, π, and Δ guarantee that ψ and σ are defined almost everywhere on X and each function has at most n values at any point. When (ψ, σ) is a simple pair, we also have the consequence that no point of X has more than n preimages under σ.

We also point out that this structure is a generalization of the structure intro-duced in Section 3 for multiple valued weighted composition operators. To see this, depending on the specific example, X might be Ω or (part of) $\partial\Omega$ and the domain Δ of Y is obtained by removing the set K as well as judiciously chosen branch cuts so as to separate the branches from one another. The compatibility conditions stated in Section 3 become the requirement that Ψ and S have the same domain Δ.

Definition 4.3. *Suppose X is a measurable space as above and suppose (ψ, σ) is a compatible pair of multiple valued (measurable) functions on X. The multiple*

valued weighted composition operator $W_{\psi,\sigma}$ on $L^2(X, d\mu)$ is the operator given by

$$(W_{\psi,\sigma}f)(x) = \sum_j \Psi_j(x)f(S_j(x))$$

Definition 4.4. *Suppose $\mathcal{H}$ is a Hilbert space of analytic functions that is a closed subspace of $L^2(X, d\mu)$ as above and suppose (ψ, σ) is a compatible pair of multiple valued (measurable) functions on X. Then the* projected multiple valued weighted composition operator $V_{\psi,\sigma}$ *is given by*

$$V_{\psi,\sigma}f = PW_{\psi,\sigma}f$$

where f is in $\mathcal{H}$ and P is the orthogonal projection of $L^2(X, d\mu)$ onto $\mathcal{H}$.

Theorem 4.5. *Suppose (ψ, σ) is a simple compatible pair of multiple valued functions the unit circle such that σ is of class C^1 on the circle and $(\sigma')^{-1}$ and ψ are in $L^\infty(\partial\mathbb{D})$. If $W_{\psi,\sigma}$ is the multiple valued weighted composition operator on $L^2(\partial\mathbb{D})$ defined by this compatible pair as above, then $W_{\psi,\sigma}$ is bounded and*

$$\|W_{\psi,\sigma}\| \le \|\psi\|_\infty \sqrt{n \, \|(\sigma')^{-1}\|_\infty}$$

where n is the multiplicity of σ.

Corollary 4.6. *Suppose (ψ, σ) is a simple compatible pair of multiple valued functions the unit circle such that σ is of class C^1 on the circle and $(\sigma')^{-1}$ and ψ are in $L^\infty(\partial\mathbb{D})$. If $V_{\psi,\sigma}$ is the projected multiple valued weighted composition operator on $H^2(\mathbb{D})$ defined by this compatible pair as above, then $V_{\psi,\sigma}$ is bounded and*

$$\|V_{\psi,\sigma}\| \le \|\psi\|_\infty \sqrt{n \, \|(\sigma')^{-1}\|_\infty}$$

where n is the multiplicity of σ.

From the definitions above, we are actually assuming that we have Δ open in $\{1, 2, \cdots, n\} \times \partial\mathbb{D}$ and a map S from Δ into $\partial\mathbb{D}$ such that S_j is C^1 on

$$\Delta_j = \{e^{i\theta}|(e^{i\theta}, j) \in \Delta\}$$

an open subset of the unit circle and that each S_j is one-to-one on Δ_j.

Proof. Suppose f is in $L^2(\partial\mathbb{D})$. Then

$$(W_{\psi,\sigma}f)(e^{i\theta}) = \sum_j \Psi_j(e^{i\theta})f(S_j(e^{i\theta}))$$

To find the norm of this function, we must integrate

$$\left| \sum_j \Psi_j(e^{i\theta})f(S_j(e^{i\theta})) \right|^2 \le \left(\sum_j |\Psi_j(e^{i\theta})||f(S_j(e^{i\theta}))| \right)^2$$

$$\le \|\psi\|_\infty^2 \left(\sum_j |f(S_j(e^{i\theta}))| \right)^2$$

Now the Cauchy-Schwartz inequality using the vector of ones in $\mathbb{C}^n$ implies

$$\left(\sum_j |f(S_j(e^{i\theta}))| \right)^2 \leq n \sum_j \left| f(S_j(e^{i\theta})) \right|^2$$

The simplicity of the pair (ψ, σ) means that we can change variables in the integral

$$\int_{\Delta_j} \left| f(S_j(e^{i\theta})) \right|^2 \frac{d\theta}{2\pi}$$

by $e^{it} = S_j(e^{i\theta})$ for e^{it} in $S_j(\Delta_j)$ to get

$$
\begin{aligned}
\int_{\Delta_j} \left| f(S_j(e^{i\theta})) \right|^2 \frac{d\theta}{2\pi} &\leq \int_{S_j(\Delta_j)} \left| f(e^{it}) \right|^2 \left| \frac{1}{S_j'(e^{i\theta(t)})} \right| \frac{dt}{2\pi} \\
&\leq \left\| \frac{1}{S_j'} \right\|_\infty \int_{S_j(\Delta_j)} \left| f(e^{it}) \right|^2 \frac{dt}{2\pi} \\
&\leq \left\| \frac{1}{S_j'} \right\|_\infty \|f\|^2.
\end{aligned}
$$

Putting this all together, we get

$$
\begin{aligned}
\int \left| \sum_j \Psi_j(e^{i\theta}) f(S_j(e^{i\theta})) \right|^2 \frac{d\theta}{2\pi} &\leq \int \left(\sum_j |\Psi_j(e^{i\theta})||f(S_j(e^{i\theta}))| \right)^2 \frac{d\theta}{2\pi} \\
&\leq \|\psi\|_\infty^2 \int \left(\sum_j |f(S_j(e^{i\theta}))| \right)^2 \frac{d\theta}{2\pi} \\
&\leq \|\psi\|_\infty^2 n \sum_j \int_{\Delta_j} \left| f(S_j(e^{i\theta})) \right|^2 \frac{d\theta}{2\pi} \\
&\leq \|\psi\|_\infty^2 n \sum_j \left\| \frac{1}{S_j'} \right\|_\infty \|f\|^2 \\
&\leq \|\psi\|_\infty^2 n \left\| \frac{1}{\sigma'} \right\|_\infty \|f\|^2
\end{aligned}
$$

Thus, we get

$$\|W_{\psi,\sigma} f\|^2 \leq \|\psi\|_\infty^2 n \left\| \frac{1}{\sigma'} \right\|_\infty \|f\|^2$$

for every f in $L^2(\partial \mathbb{D})$ which means

$$\|W_{\psi,\sigma}\| \leq \|\psi\|_\infty \sqrt{n \left\| \frac{1}{\sigma'} \right\|_\infty}$$

as we wished to prove. □

The Corollary follows trivially from the Theorem because the projection from L^2 to $H^2(\mathbb{D})$ has norm one.

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