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THE ADJOINT OF A COMPOSITION OPERATOR

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Abstract. The adjoint of a composition operator on H^2 induced by a rational function is computed explicitly as a multiple valued weighted composition operator. This computation is based on an expression for the adjoint of a composition operator on the Hardy space, and many other functional Hilbert spaces, as an integral operator. The formula for the adjoint of a composition operator on H^2 with rational symbol implies that the kernels of such operators consist of the functions in H^2 that satisfy an identity with algebraic coefficients associated with the symbol. These results generalize earlier work of Wahl and of Cowen.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk of the complex plane and $\partial\mathbb{D}$ the unit circle. If $\mathcal{H}$ is a Hilbert space of analytic functions on the unit disk and φ is an analytic function mapping the disk into itself, then for f in $\mathcal{H}$, the equation

$$C_\varphi f = f \circ \varphi$$

defines a composition operator on $\mathcal{H}$. On many common functional Hilbert spaces, conditions for boundedness and compactness of composition operators and results about their spectra and cyclicity are known. For example, if f is the Hardy Hilbert space, the Littlewood Subordination Principle [6] ensures that $C_\varphi f$ is also a Hardy function which means that C_φ acts boundedly on this space for any analytic function φ that map the unit disk into itself. In spite of this progress, many interesting and seemingly basic problems remain open. For a comprehensive treatment of some of such problems we refer to Cowen and MacCluer's book [1].

In this paper, we deal with the problem that for φ an arbitrary analytic function mapping the unit disk into itself, no satisfactory formula for the adjoint C_φ^* has been known. In this paper, we describe the adjoint C_φ^* as an integral operator and, for rational functions φ, as a new type of operator, a multiple valued weighted composition operator. The description here is self contained, but a more complete description of these new operators is contained in a separate paper [3].

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Historically, there have been two exceptions to the generalization that no satisfactory formula for the adjoint C_φ^* has been known. When φ is a linear fractional map that takes the disk into itself, then on a variety of spaces, C_φ^* has a nice expression as a product involving Toeplitz and composition operators ([2] or [1, Chapter 9]). Also, on the Hardy Hilbert space, when φ is an inner function with $\varphi(0) = 0$, C_φ is an isometry and it is easy to describe its structure explicitly, so it is easy to describe its adjoint. This can easily be combined with the formula for the adjoint for linear fractional symbols.

It has been folklore that C_φ^* acting on H^2 is given by the following nice formula (see [1], pp.322)

$$(1) \qquad (C_\varphi^* f)(z) = \int_0^{2\pi} \frac{f(e^{i\theta})}{1 - \overline{\varphi(e^{i\theta})}z} \frac{d\theta}{2\pi} \qquad (f \in H^2)$$

In the case that φ is a linear fractional map, Cowen [2] computed an expression for the adjoint C_φ^* which involved the product of Toeplitz operators and a composition operator (see also [1, Chapter 9],).

The Hardy space H^2 consists of analytic functions f on $\mathbb{D}$ for which the norm

$$\|f\|^2 = \sup_{0 \le r < 1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi}$$

is finite. Observe that the functions $f_r(e^{i\theta}) = f(re^{i\theta})$ are continuous for each $0 \le r < 1$, so they are in $L^2([0, 2\pi], d\theta/2\pi)$. Moreover, by Fatou's Theorem, any Hardy function f has radial limit at any point $e^{i\theta}$ in $\partial\mathbb{D}$ except on a set Lebesgue measure zero (see [4], for instance). Throughout this work, we will denote by $f(e^{i\theta})$ the radial limit of f at $e^{i\theta}$, that is, $f(e^{i\theta}) = \lim_{r\to 1^-} f(re^{i\theta})$.

Related to other inducing symbols, recently, Wahl [7, 8] provided a formal computation of the adjoint C_φ^* whenever φ is the particular multivalent symbol $\varphi(z) = (1-2c)z^2/(1-2cz^2)$ with $0 < c < 1/2$.

In this work we compute an explicit expression for the adjoint of a composition operator induced by any analytic function φ mapping the unit disk into itself in the Hardy space. In fact, we show that formula (1) holds for the adjoint of C_φ in H^2 without any extra assumption on the inducing symbol φ, besides $\varphi(\mathbb{D}) \subset \mathbb{D}$ of course. We proceed to state such a result:

Theorem 1.1. (A formula for C_φ^* on the Hardy space) *Let φ be an analytic function on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. Then the adjoint of the composition operator C_φ on H^2 is given by*

$$(1) \qquad (C_\varphi^* f)(z) = \int_0^{2\pi} \frac{f(e^{i\theta})}{1 - \overline{\varphi(e^{i\theta})}z} \frac{d\theta}{2\pi}$$

where f is in H^2.

In Section 2, we prove Theorem 1.1 in a rather striking way. Actually, the same idea will work to provide a formula for the adjoint of a composition operator acting on other Hilbert spaces of analytic functions on $\mathbb{D}$, such as weighted Bergman spaces.

In Section 3, after discussing the meaning of the formula (1), we express the adjoint of a composition operator with rational symbol as a multiple valued weighted composition operator. Such an expression generalizes that one found by Cowen [2] for the adjoint of a linear fractional composition operator.

2. A formula for the adjoint

In this section, we deal with the formula for the adjoint of a composition operator. The underlying idea is really simple, and it has to do with reproducing kernels. Just recall that if $\mathcal{H}$ is a functional Hilbert space of analytic functions in $\mathbb{D}$, the evaluation of functions in $\mathcal{H}$ at any point w in $\mathbb{D}$ is a bounded linear functional. So, the Riesz theorem provides a function K_w in $\mathcal{H}$, called the *reproducing kernel at* w, such that for any function f in $\mathcal{H}$

$$f(w) = \langle f, K_w\rangle_{\mathcal{H}}$$

where $\langle \cdot,\ \cdot\rangle_{\mathcal{H}}$ denotes the inner product in $\mathcal{H}$.

2.1. **A formula for C_φ^* on the Hardy space.** In order to prove Theorem 1.1, we first recall that the reproducing kernel at z in $\mathbb{D}$ in H^2 is given by

$$K_z(\zeta) = \frac{1}{1-\overline{z}\zeta}, \qquad (\zeta \in \mathbb{D})$$

On the other hand, if $\langle \cdot,\ \cdot\rangle_{H^2}$ denotes the inner product in H^2, it is well known (see [4], for instance) that for any two Hardy functions f and g, $\langle f,\, g\rangle_{H^2}$ can be computed by means of an integral of their boundary values

$$\langle f,\, g\rangle_{H^2} = \int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\,\frac{d\theta}{2\pi}$$

Now, we procceed to show formula (1). If f is in H^2, it follows

$$\begin{aligned} C_\varphi^* f(z) &= \langle C_\varphi^* f, K_z\rangle_{H^2} \\ &= \langle f,\, C_\varphi K_z\rangle_{H^2} \\ &= \int_0^{2\pi} \frac{f(e^{i\theta})}{1-\overline{\varphi(e^{i\theta})}z}\,\frac{d\theta}{2\pi}. \end{aligned}$$

Therefore, the statement in Theorem 1.1 follows. Observe that no extra hypotheses on φ has been necessary.

As a particular instance, we may compute $C_\varphi^* z^n$ for any $n \geq 0$. If D_w denotes the differential operator respect to the variable w, D_w^0 the identity operator, and D_w^n the n-th power of D_w we have the following

Corollary 2.1. *Let φ be a analytic self-map of the unit disk. If $f_n(z) = z^n$, $n \geq 0$, then*

$$(C_\varphi^* f_n)(z) = \frac{1}{n!} D_w^n \left(\frac{1}{1 - \overline{\varphi(\overline{w})} z} \right) \Big|_{w=0}$$

Just observe that the map $\overline{\varphi(\overline{w})}$ is analytic on $\mathbb{D}$, so the expression in Corollary 2.1 makes sense.

2.2. **A formula for C_φ^* on weighted Bergman spaces.** For $\alpha > -1$, recall that the weighted Bergman space $\mathcal{A}_\alpha^2$ consists of functions f analytic on $\mathbb{D}$ for which the norm

$$\|f\|_\alpha^2 = \int_{\mathbb{D}} |f(w)|^2 (1 - |w|^2)^\alpha \, dA(w)$$

is finite. Here, $dA(z) = \dfrac{1}{\pi} \, dx \, dy$ denotes the normalized Lebesgue area measure on $\mathbb{D}$. Observe that $\mathcal{A}_\alpha^2$ are Hilbert spaces of analytic functions on $\mathbb{D}$. In addition, $\mathcal{A}_\alpha^2$ turns out to be the classical Bergman space $\mathcal{A}^2$ when $\alpha = 0$.

A little computation shows that the reproducing kernel functions in $\mathcal{A}_\alpha^2$ are

$$K_z(w) = \frac{\alpha + 1}{(1 - \overline{z} w)^{\alpha + 2}} \qquad (z \in \mathbb{D})$$

Therefore, the same argument as before yields

Theorem 2.2. (A formula for C_φ^* on weighted Bergman spaces) *Let φ be an analytic function on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. Then the adjoint of the composition operator C_φ on $\mathcal{A}_\alpha^2$ is given by*

$$(C_\varphi^* f)(z) = \int_{\mathbb{D}} \frac{(\alpha + 1) f(w)}{(1 - \overline{\varphi(w)} z)^{\alpha + 2}} (1 - |w|^2)^\alpha \, dA(w) \tag{2}$$

where f is in $\mathcal{A}_\alpha^2$.

As before, we may deduce C_φ^* acting on the orthogonal system $\{z^n\}_{n=0}^\infty$ in $\mathcal{A}_\alpha^2$. A little computation shows that $\|z^n\|_\alpha^2 = n!/(\alpha + 1)(\alpha + 2) \cdots (\alpha + n + 1)$ (see also [1, Chapter 3]). Therefore, we have the following result.

Corollary 2.3. *Let φ be a analytic self-map of the unit disk. If $f_n(z) = z^n$, $n \geq 0$, then in the weighted Bergman space $\mathcal{A}_\alpha^2$ it holds the following*

$$(C_\varphi^* f_n)(z) = \frac{1}{(\alpha + 2) \cdots (\alpha + n + 1)} D_w^n \left(\frac{1}{(1 - \overline{\varphi(\overline{w})} z)^{\alpha + 2}} \right) \Big|_{w=0}$$

Remark 2.4. *(A formula for C_φ^* on weighted Dirichlet spaces).* For $\alpha > -1$, we recall that the weighted Dirichlet space $\mathcal{D}_\alpha$ is the collection of analytic functions f on $\mathbb{D}$ for which the complex derivative f' belongs to the weighted Bergman space $\mathcal{A}_\alpha^2$. Each of these spaces can be normed in several different ways, each way giving equivalent norms, turning the weighted Dirichlet spaces into Hilbert spaces of analytic functions. Since the adjoint of an operator in a Hilbert space depends strongly on the norm considered, the different definitions of norm give different expressions for the adjoint, but one can deduce the expression for C_φ^* once the norm is chosen and the corresponding reproducing kernels K_w are determined.

3. C_φ^* AS A MULTIPLE VALUED WEIGHTED COMPOSITION OPERATOR

In this section, we introduce weighted composition operators with multiple valued symbols to describe the adjoint of a composition operator induced by a rational function acting on the Hardy space. Basic properties of such operators acting on functional Banach spaces of analytic functions are studied in [3]. Throughout this section, we collect some of their basic facts to make the paper self-contained. Nevertheless, we begin by considering an example which sheds some light on the spirit behind that description.

3.1. **An example.** Let $\varphi(z) = (z^2+z)/2$ for z in $\mathbb{D}$. It is clear that φ takes $\mathbb{D}$ into itself and induces a bounded composition operator on H^2. By means of formula (1), we have

$$\begin{aligned} C_\varphi^* f(z) &= \frac{1}{\pi}\int_0^{2\pi} \frac{f(e^{i\theta})}{2-(e^{-2i\theta}+e^{-i\theta})z}\,d\theta \\ &= \frac{1}{\pi}\int_0^{2\pi} \frac{e^{2i\theta}f(e^{i\theta})}{2e^{2i\theta}-(1+e^{i\theta})z}\,d\theta \\ &= \frac{1}{\pi i}\int_{\partial\mathbb{D}} \frac{\zeta f(\zeta)}{2\zeta^2-(1+\zeta)z}\,d\zeta, \end{aligned} \tag{3}$$

where in last line the change of variable $\zeta = e^{i\theta}$ has been carried out. Now, observe that $2\zeta^2-(1+\zeta)z = 2(\zeta-\zeta_1)(\zeta-\zeta_2)$, where

$$\zeta_1 = \frac{z+\sqrt{z^2+8z}}{4} \qquad \text{and} \qquad \zeta_2 = \frac{z-\sqrt{z^2+8z}}{4}.$$

If $z \neq 0$, upon applying the Residue Theorem, it follows that the integral in (3) is equal to

$$\frac{1}{\zeta_1-\zeta_2}\left(\zeta_1 f(\zeta_1) - \zeta_2 f(\zeta_2)\right)$$

or equivalently, we have

$$C_\varphi^* f(z) = \frac{z+\sqrt{z^2+8z}}{2\sqrt{z^2+8z}}\, f\left(\frac{z+\sqrt{z^2+8z}}{4}\right) - \frac{z-\sqrt{z^2+8z}}{2\sqrt{z^2+8z}}\, f\left(\frac{z-\sqrt{z^2+8z}}{4}\right).$$

First, recall that z is different from zero in the above expression. In addition, formally, we may express C_φ^* by

$$C_\varphi^* f(z) = \sum \psi(z)(f\circ\sigma)(z), \tag{4}$$

where $\psi(z) = \left(z \pm \sqrt{z^2+8z}\right)/2\sqrt{z^2+8z}$, $\sigma(z) = (z\pm\sqrt{z^2+8z})/4$ and the sum is taken over all the branches of ψ and σ combined, roughly speaking, in a special way. The important point to note here is that we are not allowed to combine all the branches of ψ and σ. Although Equation (4) is a formal expression, the principal significance of this example is the relationship between the adjoint of a composition operator and certain weighted composition operators. (Compare Wahl's work [7, 8].) In what follows, we formalize such a relation in terms of multiple valued weighted composition operators.

3.2. **Multiple valued weighted composition operators.** First, we introduce the concept of a *compatible pair of multiple valued functions* on a domain Ω in the complex plane $\mathbb{C}$ (see also [3]).

Definition 3.1. *Let $\Omega \subset \mathbb{C}$ be a domain and z_0 a point of Ω, the* base point*. Let K be a finite set in Ω that does not include z_0. Suppose ψ and σ are functions analytic in a simply-connected neighborhood of z_0 in $\Omega \setminus K$ and suppose they are arbitrarily continuable in $\Omega \setminus K$. We say (ψ, σ) is a* compatible pair of multiple valued functions on Ω *if for any path γ in $\Omega \setminus K$ along which the continuation of σ yields the same branch as at the beginning, it is also the case that continuation of ψ along γ yields the same branch as at the beginning.*

The definition above apparently depends on the base point z_0 and the finite set K and on the functions ψ and σ defined in a neighborhood of z_0 identified in the statement. However, informally, we regard ψ and σ as the names of the multiple valued functions defined in $\Omega \setminus K$ and the intent of the definition is to connect each branch of σ at a point with a specific branch of ψ at that point. Because this association is independent of the base point z_0 and because the smallest allowable K for given Ω and σ follow from the definition of σ, we do not usually include the base point and the set K in the description, but just say (ψ, σ) is a compatible pair of multiple valued functions on X.

Note that it is a consequence of the definition that if (ψ, σ) is a compatible pair on Ω then the number of branches of ψ at a point is a divisor of the number of branches of σ at that point. The number of branches of σ at each point will be called the cardinality of the pair or sometimes the pair of integers (number of branches of ψ, number of branches of σ) will be called the cardinality of the pair.

Also note that if σ has a removable singularity at a point ξ in K and if ψ is bounded in a punctured neighborhood of ξ, then ψ has a removable singularity at ξ as well: because each branch of σ is single valued in a neighborhood of ξ, the fact that each branch of σ is associated with a particular branch of ψ means that ψ is also single valued in a neighborhood of ξ.

Definition 3.2. *Suppose Ω is a domain in the complex plane, and K is a finite subset of Ω. Suppose, further, that σ is an n-valued analytic function that is arbitrarily continuable in $\Omega \setminus K$ and takes values in Ω. Assume that ψ is an m-valued, where m divides n, bounded analytic function that is arbitrarily continuable in $\Omega \setminus K$. The multiple valued weighted composition operator $W_{\psi,\sigma}$ on $\mathcal{H}$ is the operator defined by*

$$W_{\psi,\sigma} f(z) = \sum \psi(z) f(\sigma(z))$$

for f in $\mathcal{H}$ and where the sum is taken over all the branches of the pair (ψ, σ) for z in $\Omega \setminus K$.

Multiple valued weighted composition operators are bounded operators acting on the Hardy space $H^2(\mathbb{D})$ (see [3, Theorem 2.5]). Moreover, in the same way as composition operators or multiplication operators, they can be characterized in terms of their action on the reproducing kernels (see [3, Theorem 2.7]).

The goal of the following result is to generalize, in some sense, the expression for the adjoint of composition operators induced by linear fractional maps obtained in [2] to a similar expression for the adjoint of composition operators induced by rational maps. In fact, such an expression involves multiple valued weighted composition operators.

Theorem 3.3. *Let φ be a rational map taking $\mathbb{D}$ into itself. Let $\widetilde{\varphi^{-1}}$ denote the multiple valued algebraic function defined by $\widetilde{\varphi^{-1}}(z) = \overline{\varphi^{-1}(1/\overline{z})}$. Then, for any f in H^2*

$$C_\varphi^* f(z) = BW_{\psi,\,\sigma} f(z)$$

where B is the backward shift operator and $W_{\psi,\,\sigma}$ is the multiple valued weighted composition operator induced by $\sigma = 1/\widetilde{\varphi^{-1}}$ and $\psi = (\widetilde{\varphi^{-1}})'/\widetilde{\varphi^{-1}}$.

Before proceeding further, observe that σ is a multiple valued function that takes $\mathbb{D}$ into itself since so does the rational map φ. In addition, no branch of $\widetilde{\varphi^{-1}}$ vanishes in $\mathbb{D}$, so $W_{\psi,\sigma}$ is well defined on $\mathbb{D}$.

Proof of Theorem 3.3. In order to get an expression for C_φ^*, let f and g be polynomials. Let $\widetilde{g}$ be the holomorphic function in $\{z : \ |z| > 1\}$ defined by $\widetilde{g}(z) = \overline{g(1/\overline{z})}$. It holds that if g has non-tangential limit $g(\zeta)$ at ζ in $\partial\mathbb{D}$, then so does $\widetilde{g}$ and $\widetilde{g}(\zeta) = \overline{g(\zeta)}$. Then,

$$\begin{aligned} \langle f, C_\varphi^* g\rangle_{H^2} &= \frac{1}{2\pi}\int_0^{2\pi} f(\varphi(e^{i\theta}))\overline{g(e^{i\theta})}\, d\theta \\ &= \int_{\partial\mathbb{D}} f(\varphi(\zeta))\widetilde{g}(\zeta)\, \frac{d\zeta}{\zeta} \end{aligned}$$

Let $\varphi^{-1}(\mathbb{D})$ denote the component in $\mathbb{C}$ which contains $\mathbb{D}$. Observe that since φ has no poles in $\varphi^{-1}(\mathbb{D})$, and $\tilde{g}$ is holomorphic $\{z : \ |z| > 1\}$, Cauchy's Theorem yields

$$\langle f, C_\varphi^* g\rangle_{H^2} = \int_{\partial\varphi^{-1}(\mathbb{D})} f(\varphi(\zeta))\widetilde{g}(\zeta)\, \frac{d\zeta}{\zeta} \tag{5}$$

Now, since φ is a rational map, there exists a positive integer N and N arcs of curve $\Gamma_j \subset \partial\varphi^{-1}(\mathbb{D})$ such that

$$\partial\varphi^{-1}(\mathbb{D}) = \bigcup_{j=1}^{N} \Gamma_j$$

$\Gamma_j \cap \Gamma_k = \emptyset$ for $j \neq k$ and, for each $j = 1, \cdots, N$ the arc of curve Γ_j is mapped onto $\partial\mathbb{D}$ by φ. Thus, the integral in (5) is equal to

$$\sum_{j=1}^{N} \int_{\Gamma_j} f(\varphi(\zeta))\widetilde{g}(\zeta)\,\frac{d\zeta}{\zeta} \tag{6}$$

At this point, we may proceed with the change variables $\varphi(\xi) = \eta$ in each of the integrals involved in the sum in (6), since φ takes Γ_j bijectively onto $\partial\mathbb{D}$. Once again, abusing of notation, we will write φ^{-1} for all branches φ_j^{-1} of φ^{-1}. Then, it follows

$$\langle f, C_\varphi^* g\rangle_{\mathcal{H}^2(\mathbb{D})} \quad = \int_{\partial\mathbb{D}} \eta\, f(\eta) \left(\sum_{j=1}^{N} \frac{\widetilde{g}(\varphi^{-1}(\eta))}{\varphi^{-1}(\eta)\,\varphi'(\varphi^{-1}(\eta))} \right) \frac{d\eta}{\eta}$$

Now, observe that whenever $\eta \in \partial\mathbb{D}$ it holds

$$\widetilde{g}(\varphi^{-1}(\eta)) = \overline{g\left(\frac{1}{\widetilde{\varphi^{-1}}(\eta)}\right)}$$

Therefore, a little computation shows that

$$\langle f, C_\varphi^* g\rangle_{\mathcal{H}^2(\mathbb{D})} = \int_{\partial\mathbb{D}} \eta\, f(\eta)\, \overline{\left(\sum \frac{\widetilde{(\varphi^{-1})'}(\eta)}{\widetilde{\varphi^{-1}}(\eta)}\;\; g\left(\frac{1}{\widetilde{\varphi^{-1}}(\eta)}\right)\right)}\, \frac{d\eta}{\eta}$$

which is the desired expression.

□

This gives a simple condition for the functions in the kernel of C_φ^* when φ is a rational function.

Corollary 3.4. *Let φ be a rational map taking $\mathbb{D}$ into itself and let ψ and σ be defined as in the statement of Theorem 3.3. Then f in H^2 is in the kernel of C_φ^* if and only if*

$$\sum \psi(z) f(\sigma(z)) = \sum \psi(0) f(\sigma(0))$$

for all z in $\mathbb{D}$.

Proof of Corollary 3.4. The function f in H^2 is in the kernel of C_φ^* if and only if $BW_{\psi,\sigma} f$ is the zero function. Since the kernel of the backard shift operator B is the subspace of constant functions, this means f is in the kernel of C_φ^* if and only if $W_{\psi,\sigma} f$ is a constant function. That is, the value of $\sum \psi(z) f(\sigma(z))$ is the same for every z in $\mathbb{D}$ as it is at $z = 0$.

□

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