ON THE CENTRALITY OF LINEAR ALGEBRA IN THE CURRICULUM

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While this is an opportunity for me to lay out my "secrets" for successful teaching, I have too few secrets and they are too well known for me to talk for more than a few minutes about them.

The first step for a person to take toward good teaching is to recognize that the pursuit of teaching excellence is a worthy goal. Since I believe that good teaching is important, I'm very glad that today, this first step is getting easier for many people to make. Having decided that teaching well is important, the next step is to think about the issues involved and to educate oneself about the things that work for other people. I'm very grateful to my colleagues who have modeled good teaching, answered my questions about approaches to a lesson or a subject, and encouraged me in my efforts to teach more effectively. Some of the people who deserve my thanks are Bill Fishback and Harold Hanes at Earlham College and Guershon Harel, Jim McClure, J. J. Price, and Bob Zink at Purdue. I have learned a great deal about teaching mathematics from talking with them, but I have learned even more about teaching from talking with my wife Janice, who is an exceptional Spanish teacher. Finally, it is important to know the students, to talk with them and, especially, to listen to them. Get to know their names, get them to talk in class and ask questions, and get them to talk outside of class, too. Students will tell you, although not always directly, what they don't understand and then you can help them develop their own answers to their questions.

If you know the students' names, you can say hello to them in the hall and you can ask them, by name, to answer a question in class. While this is intimidating, if you ask every student in the class a question every week or two, they will quickly learn that they aren't being picked on. Generally, I don't take "I don't know" as an answer to a question, it just leads me to ask a simpler related question that will help them discover the answer to the question I originally asked. When students are in the habit of speaking in class, they will more readily ask questions and if you routinely ask them questions, they are more likely to ask questions of you before you ask them the question they can't answer. Of course, how you handle questions is critical. You might be fortunate enough that all their questions are insightful and lead you on to the next topic, but don't count on it. You must take pains to regard every question as serious and deserving of a thoughtful reply. Students will only ask questions if they are reasonably comfortable doing so and your indication that any question is fair will help them be comfortable about asking questions even when they can't tell which questions are "dumb". But just because someone

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asks a question doesn't mean I give them a straight and simple reply! I'm much more likely to regard a question as a starting point for a discussion of what they understand and don't understand about the situation.

Instead of talking more about my secrets, I want to talk at some length about the role of linear algebra in the curriculum and the opportunities it presents for teaching. A good friend of mine tells me that teaching differential equations is much more rewarding and interesting; she is probably right, for her. For me, though, linear algebra has become the focus of my instructional work and having that focus has been critical to my development as a teacher; I urge you to find the venues that are right for you to develop your teaching skills and that will enable you to make the strongest contributions in your institution.

1. A Little History

I taught linear algebra the first semester I was in a college classroom and most semesters since then. At first this was accidental, but later, as I began to consider the curricular issues involved, I realized that linear algebra plays a pivotal role in the curriculum both for majors and mathematically oriented non-majors, so I wanted to teach it frequently.

There is a tendency to believe that the structure of mathematics and, with the exception of calculus reform, the college mathematics curriculum have remained the same for a very long time. This is far from true. In fact, linear algebra, as we know it today, has existed a comparatively short time.

My first realization of the changes that have taken place came from seeing the MAA film "Who Killed Determinants?" (done by Kenneth O. May in the 1960's, even before Sheldon Axler [1] decided it was a good idea they be killed). May documented how determinants flourished in the 19th century with its connections to the study of invariants and how the study of determinants developed into the linear algebra we know today, where determinants are far from central. Linear algebra did not really come to be recognized as a subject until the 1930's. Particularly influential in this process were the book of B. L. van der Waerden [9] from 1930–31 and the book of Garrett Birkhoff and Saunders MacLane [2] of 1941. Both were on "Modern Algebra," but included chapters on linear algebra. Historian Jean-Luc Dorier [5] regards Paul Halmos' book [6] "Finite Dimensional Vector Spaces," first published in 1942, as the first book about linear algebra written for undergraduates. This is all much more recent than I would have guessed a few months ago!

In 1936–37 at Harvard, Birkhoff taught an algebra course that included an axiomatic treatment of vector spaces over a field and linear transformations on finite dimensional vector spaces and in '39–'40 MacLane taught the same course [8, page 295]. In preparation for this talk, I looked at catalogs from several colleges and universities to find out when their first undergraduate courses in linear algebra were taught. The separate linear algebra course became a standard part of the college mathematics curriculum in the United States in the 1950's and '60's and some colleges and universities were still adding the course in the early 1970's. It appears that the linear algebra course I had in 1965 at Indiana University was one of the first times it was offered there as a regular course although, at the time, I thought math majors had been taking it for decades. The catalogs make it clear that linear algebra courses had been split away from the abstract algebra courses that had developed earlier. This was reflected in the very abstract nature of the courses many of us took then: indeed, I could prove theorems on determinants of linear transformations on an abstract vector space but would have had difficulty in finding the determinant or inverse of a 4 by 4 matrix!

Thus, in the past 40 years or so, the linear algebra course has come into being as an abstract course for serious majors, has been revised into a first "intro to proof" and "intro to abstract mathematics" course for all math majors, and in many places has now become a sophomore matrix-oriented course for a wide variety of majors.

2. The Reform of the Linear Algebra Course

Why am I telling you this? I want you to realize that (regardless of the prevailing attitudes in your department) linear algebra has not "always" been done the way it is now, to suggest that we are in the middle of a "reform", and to use the history of the reform so far to point out where I think we are and should be going.

The first step is try to understand the developments up to now. I believe that the first courses grew out of the general axiomatic approach to mathematics that was common at that time. Historian Gregory Moore [8] regards the axiomatization of abstract vector spaces to have been completed in the 1920's and many areas of mathematics had their foundations developed in the first third of the century. I think the success of the axiomatic method in this and related algebraic areas, as well as the basic and important mathematical content, contributed to abstract algebra and linear algebra being given a prominent place in the curriculum first for serious majors then for all math majors.

But the more recent phase of the reform has a different origin: I believe it is due to the development and widespread use of the computer in areas that apply mathematics. Surely engineers have known for more than a century that many problems could be modeled as systems of linear equations or as eigenvalue problems. But what would be the point? Even in the 1950's, few engineers could hope to solve a system of 100 equations in 100 unknowns; linear algebra was really irrelevant! By the '70's engineers were beginning to use computers to solve practical problems using linear algebra. For example, in 1974, a graduate student friend studying civil engineering and working on modeling vibrations in buildings caused by earthquakes asked me how he could find the eigenvalues of a 200×200 matrix that were close to 12. (Unfortunately, at the time, I had no clue – the best advice I had to offer was to find all 200 and check which were closest to 12; I know better now!) In the past two decades, the applications of linear algebra to real world problems have mushroomed. The computer software Matlab provides a good example: it is among the most popular in engineering applications and at its core it treats every problem as a linear algebra problem. Suddenly students from all over the university are being advised to take a linear algebra course. The influx of these students with their different interests and, with the higher percentage of the population going on to college, the influx of students who are mathematically not as well prepared have forced many colleges and universities to change from courses dominated by proofs of theorems about abstract vector spaces to courses emphasizing matrix computations and the theory to support them.

3. The Role of the Computer in the Classroom

The change in the audience for our linear algebra courses creates the necessity, and the opportunity, to re-examine our approach to teaching the subject. After all, the essence of teaching is to help students learn the material that they need and want to learn: with different students wanting to learn different material, we should expect to change our teaching.

In thinking about teaching this subject, the first realization that we must come to is that linear algebra is incredibly useful in the modern world, probably more useful than any other college level mathematics with the possible exception of calculus. Several of my former students have told me that linear algebra was the most useful math course they took in college and can give specific examples of why they say that. Students who are in the courses now believe this is the case and most new linear algebra books include applications that look to students like they are non-trivial. While I don't believe this should force us to teach the applications, I do believe it necessitates our being conscious of the applicability and the change that requires in our teaching style. I don't believe it forces us to abandon theory, but it should encourage us to see the role of the theory in the subject as it is applied.

The second realization we must come to is that no serious application of linear algebra happens without a computer. This must change the nature of the course; I believe it strongly argues for including computation as a part of the course. Fortunately, many calculators can do all the computations that arise in a first course and software such as Matlab, Maple, and Mathematica can do all that and more. We need to be at least vaguely conscious of the ways computers do linear algebra and how that affects the way we approach the subject. For example, the standard approach to eigenvectors and eigenvalues, and for years the only one I knew, was to find the characteristic polynomial of the matrix, find the roots of the polynomial, and solve the eigenvector equations for each eigenvalue. This approach is hopeless for practical sized matrices! Indeed, for large matrices, it is difficult to do even the first step of finding the characteristic polynomial. Nevertheless, the relationship between the characteristic polynomial and the eigenvalues is still important for students to understand and the QR-algorithm, a numerical approach for finding eigenvectors and eigenvalues, is not appropriate material for a first course in linear algebra. But, in spite of this, our students should not leave our courses thinking the characteristic polynomial is the only way to find the eigenvalues of a matrix.

What I am really arguing for is an effective integration of computing into the classroom. Although I don't know what that is in every circumstance, leaving the computer out of the classroom is certainly not the most effective approach. Many faculty around the country are working on how to incorporate computing effectively and the answers that are being developed are sure to change the way our classrooms will look, how we will teach, and ultimately, how our students will learn. One point that deserves mention is that math majors tend to be less computer literate than some other majors, a disadvantage in the 1990's. If we integrate computing into the math majors' courses, they will graduate with more confidence and more useful computer experience, and I believe we will therefore be able to attract more majors.

At the surface level, many faculty agree with the point that after the basic hand calculation techniques have been mastered, having students use a machine to do the arithmetic is very helpful. For one thing, the students can concentrate on this week's ideas instead of trying to get the arithmetic right in the solution of the relevant linear system.

At a deeper level, though, if the instructor is armed with a computer and display device so that the class can see her or his work, the instructor can do things that are not otherwise possible in class. For instance, in working an example, I like to ask the ask the class how to approach the problem. If I am working strictly with a blackboard and prepared-in-advance calculations, when a student proposes an inappropriate approach to the problem, I will not have those calculations preprepared and will be unwilling to take class time to do them at the blackboard: I am left explaining, perhaps unconvincingly, why the suggested approach is inappropriate. On the other hand, armed with a computer, I am usually willing to do exactly what the students tell me to do: when the misguided attempt fizzles, they see it first hand rather than simply taking my word for it, and they are motivated to participate in a second attempt. One of a teacher's roles is to demonstrate thought processes. Too many times, a lecturer seems to be omniscient, always knowing exactly how to do every problem. Students aren't like that – they occasionally make mistakes and they need to learn how to recognize them as mistakes and recover. You and I know that we are not perfect in our offices, but the students don't. We see when we have made a mistake and we have learned how to start again with a new approach – our students will benefit from seeing us recover from a failed attempt.

In addition, I am more willing to check the result of a calculation if I have a computer because it is fast. For example, if the question asks us to split a vector z into two pieces $z = w + u$ in such a way that w is in the subspace M and u is orthogonal to it, checking provides a mental review of the problem as we are punching the keys to check that $z = w + u$, w is in M, and u is orthogonal to M, and they can rethink what the conditions mean. My colleague Jim McClure likes to start a calculation and then ask what will happen when he pushes the "return" key. He finds students are more willing to participate when he is using the computer than if he does the computing at the board. I suspect students can imagine themselves in the situation looking at the computer screen more easily than across the desk from the instructor.

Some faculty worry that using computers in a math course will turn the students into unthinking button-pushers. While this could be true in some circumstances, it is easy to avoid in linear algebra. Indeed, I think the computer can be used to motivate learning the theory and to reinforce the concepts. I believe that in linear algebra, more than in any other elementary college mathematics course, the theory plays an essential role in computations and that use of the computer can make this evident. A good example is solution of linear systems. The theorem engineers call the "Principle of Superposition" says that every solution of a linear system can be written as the sum of a particular solution of the system and some solution of the related homogeneous system. In the past, I have found this theorem to be difficult for students to understand and appreciate. Matlab has a command "\" that gives one solution to any system, namely the least squares solution, and another command "null" that gives an orthonormal basis for the nullspace of a matrix. The software provides both a reason to understand the theory and a mechanism for using the theorem to easily find the general solution of a system of equations.

In addition to examples of solving problems, computers make possible "gee whiz" classroom demonstrations that can stimulate the students' geometric intuition. I regularly use Matlab "movies" to demonstrate linear transformations of the plane and their eigenvectors. I also have used similar movies to demonstrate the geometric meaning of the Singular Value Decomposition in my graduate class for engineering students. Right now, Roger Lautzenheiser, of Rose-Hulman Institute of Technology, and his student Brad North are putting the finishing touches on a demonstration package for linear algebra courses. It runs under Matlab and has a nice enough interface that students can play with the geometry of linear transformations and their ranges and nullspaces and can see the rank–nullity theorem in action. Certainly, their demonstration at last fall's Indiana Section meeting elicited lots of "gee whiz's"! I understand the package will be publicly available soon and that it will be free.

In addition, the computer makes it possible to ask questions involving theoretical issues that are arithmetically too complicated to ask a student to do with pencil and paper. For example, many students believe that every subspace has a special, preferred basis and do not really understand the implications of the fact that every subspace has infinitely many bases. A question I like to ask students to do on the computer is the following:

> John and Mary are taking linear algebra. One of the problems in their homework assignment was to find the nullspace of the 4×5 matrix A. John's answer was that the nullspace is spanned by $(-2, -2, 0, 2, -6)$, $(1, 5, 4, -3, 11)$, $(3, 5, 2, -4, 13)$, and $(0, -2, -2, 1, -4)$.
>
> Mary's answer was that the nullspace is spanned by $(1, 1, 0, -1, 3)$, $(-2, 0, 2, 1, -2)$, and $(-1, 3, 4, 1, 5)$. Are their answers consistent with each other?

I think the question brings out the idea that there can be more than one spanning set for a subspace in a context that is meaningful to the students and I think it requires students to confront issues that are difficult for them like Which vectors are in a subspace? When are two subspaces the same?

One approach to this problem is to try to write each of John's vectors as a linear combination of Mary's and vice versa. The definitions and first theorems about spanning sets say that if each of John's vectors is a linear combination of Mary's vectors, and each of Mary's vectors is a linear combination of John's vectors, then the subspaces they have described are the same, otherwise they are not. By hand, this would be incredibly tedious, but with a machine, it is a tolerable approach. A more sophisticated solution is to find the rank of the 5×4 matrix whose columns are John's vectors and then find the rank of the 5×7 matrix whose columns are John's and Mary's vectors both. Matlab can find the rank of a matrix in a single command, so this is very easy on the computer. On the other hand, this approach requires students to really understand how the column vectors are connected with the range of a matrix and how the dimensions of nested subspaces are related to equality of the subspaces.

Many students do not have a good idea of how to start this problem and many will try some approach and need to start over when they realize that it will not work. The fact that the students have access to a machine makes me willing to ask questions like this that may require several false starts and a lot of arithmetic. I would typically assign this problem as homework and after the class has struggled with the problem, talk about both approaches, using the computer to demonstrate.

Reform in linear algebra is healthy and moving forward. In 1993, the Linear Algebra Curriculum Study Group published [3] a set of recommendations for appropriate course syllabuses for various kinds of classes. While you may not agree with all of their recommendations, you will surely find their work stimulating and thought provoking as you design your own course. The ATLAST project has developed and published a set of computer projects [7] suitable for linear algebra classes and the MAA will soon publish a book [4] of projects useful in teaching linear algebra as well. Many colleges and universities are introducing a second course in linear algebra because they recognize the importance of the subject and the inadequacy of their first course in meeting the diverse needs of the students. In addition, faculty all over the country are rethinking the place of linear algebra in their curriculum and the best approaches for teaching it. Indeed, there were three sessions at the San Diego meeting on innovations in teaching linear algebra.

To summarize, I believe that linear algebra deserves a central place in the curriculum of math majors, and other students as well, because it widely applicable, because it is a subject where students can see, even without axiomatics, the development of a substantial mathematical theory, because it is a subject that provides the opportunity for students to see the role of that theory in doing computations and applying mathematics, and because it provides a vital arena where students can see the interaction of mathematics and machine computation. I believe that the integration of computation and theoretical mathematics is so natural in linear algebra that students (and faculty!) can use their experience with linear algebra as a starting point for seeking similar integration in other mathematical areas. Linear algebra provides a course that is full of ideas, with material that is rewarding to learn, and to teach, and is a subject where both student and teacher can be challenged to their best performance. Finally, I don't think students arriving as Freshmen know how to learn, certainly they don't know how to learn mathematics. Students need to learn how to integrate a theoretical and computational understanding of mathematics. Learning linear algebra can help them do that: Students who have learned how to learn linear algebra have learned how to learn mathematics!

I believe teaching is important and linear algebra is one of my favorite places to use and improve my teaching skills. I am indeed grateful to be recognized by the MAA for my efforts in teaching.

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